A Foundation for Calculus: Exploring Mathematical Misconceptions Related to the Teaching and Learning of Tangent Lines

Mark D. Hogue & Dominic Scarcelli, *Slippery Rock University of Pennsylvania*

Abstract: *Misconceptions of mathematical content are formidable barriers to current and future learning. If these misconceptions go unnoticed they can stay with the student for years, impacting future learning. Tangent lines are a concept that students find particularly challenging, resulting in a variety of these kinds of misconceptions. This paper intends to describe a general plan for reducing student misconceptions in general, to provide some information on the nature of common student misconceptions about tangent lines, and to give examples of how these misconceptions can be prevented, identified, and corrected.*

Keywords: *calculus, misconceptions, tangent lines*

1 Introduction

Within the field of mathematics education, teachers constantly encounter errors in students' work and thinking. Yet, the errors that are made in the context of a mathematics classroom are not equal. There is a significant difference between careless arithmetic or procedural mistakes and serious conceptual errors. The former hints at nothing except for, possibly, a lack of practice, while the latter may point to serious errors in student understanding of the concept, revealing a need to redesign the way the topic is presented to students. We have intended this paper to serve as an aid for teachers of mathematics through calculus who deal with the concept of tangency, but we would regard it as particularly useful for teachers of calculus and pre-calculus.

Teaching tangent lines to students can be a demanding and challenging process for a several reasons. In the first place, individual tangent lines can satisfy a variety of properties which seem to contradict each other (Winicki-Landman & Leikin, 2000). For example, one collection of tangent lines might intersect a function at only a single point (refer to Figures 1 and 2), while another set may contain tangent lines with multiple intersection points (refer to Figure 3).

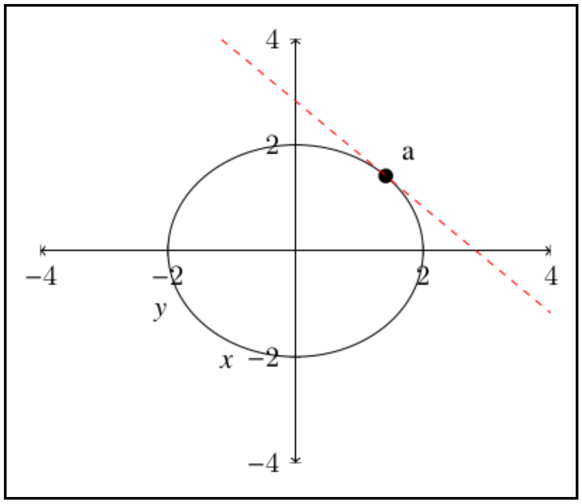

Fig. 1: *A line is tangent to a circle if and only if it intersects the circle at a single point.*

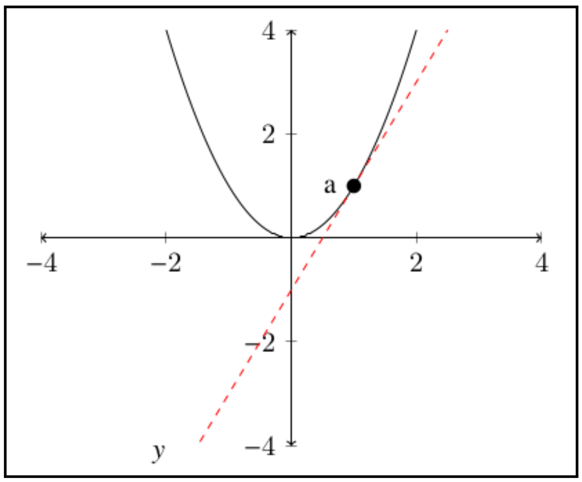

Fig. 2: *The tangent line to a parabola intersects the parabola at exactly one point and leaves the parabola entirely on one side of the line.*

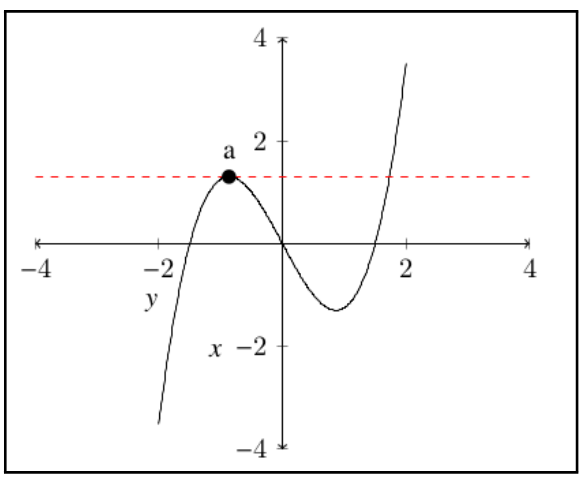

Fig. 3: *The tangent line given below intersects the function at multiple points.*

1.1 Prior Ideas about Tangent Lines

As previous studies have shown, students enter the classroom with a multitude of prior ideas about tangent lines from geometry or algebra (Biza, Christou, & Zachariades, 2008; Vincent, Sealey & Engelke, 2015; Winicki-Landman & Leikin, 2000). In geometry, students may have encountered tangent lines in reference to circles, where a tangent is often defined as a line which intersects the circle exactly one time (Figure 1) (Biza, Christou, & Zachariades, 2008; Winicki-Landman & Leikin, 2000). Similarly, in algebra, students may have encountered tangent lines when studying parabolas. Here, the tangent is often defined as a line which intersects the parabola at exactly one point and which leaves the parabola entirely in one half plane (Figure 2) (Biza, Christou, & Zachariades, 2008; Winicki-Landman & Leikin, 2000). A combination of these factors can lead to a situation where a variety of misconceptions may take root.

1.2 Concept Image

Of particular importance to our discussion is Tall and Vinner's theory of the concept image, described as "the total cognitive structure that is associated with the concept, which includes all mental pictures and associated properties and processes" (Tall & Vinner, 1981). When some portion of a student's concept image is in contradiction with the actual concept, Tall and Vinner note this as a "potential conflict factor" (Tall & Vinner, 1981). This potential conflict factor becomes actualized as a cognitive conflict factor if the associated cognitive dissonance actually causes the student to enter into cognitive conflict.

A student's personal concept definition is a form of words which they use to specify that concept. The formal concept definition, on the other hand, is the definition of the concept that has gained general acceptance in the greater mathematical community (Tall & Vinner, 1981). For a student who holds misconceptions of a given concept, cognitive conflict may not be possible if the actual concept definition is not a part of a student's concept image, noted by Tall and Vinner:

> Such factors can seriously impede the learning of a formal theory, for they cannot become actual cognitive conflict factors unless the formal concept definition develops a concept image which can then yield a cognitive conflict. Students having such a potential conflict factor in their concept image may be secure in their own interpretations of the notions concerned and simply regard the formal theory as inoperative and superfluous (Tall & Vinner, 1981).

1.3 The Importance of Formal Definition

Mindful of these implications, it becomes incredibly important to introduce students to the formal definition of a concept as soon as possible. Without a formal definition, students will begin to develop certain conceptions which very well may be incorrect. Generally, as students have yet to be introduced to the formal definition of a particular concept, they have no way to test their conceptions within the formal theory. As a result, they will likely be entirely unaware that they do not properly understand the given concept. Furthermore, as we will see below, the formal definition of a concept is a powerful tool for the teacher to use in helping students with their misconceptions.

2 Student Misconceptions

Efforts to prevent and remedy student misconceptions can be separated into three distinct groups of actions: prevention measures, identification measures, and correction measures. Prevention measures are actions which are taken to prevent students from developing certain misconceptions and generally take place during the main body of a lesson. A simple example of this comes from geometry, where students often calculate the area of the rhombus as the product of the lengths of two adjacent sides, rather than as the length of the base times the height. The area of a rhombus is an example of Cavalieri's principle, which is a rich source for visual and physical demonstrations. For example, as a preventative measure, teachers can show how a rectangle can be broken into smaller and smaller pieces and how these pieces can be slid so that the whole shape starts to resemble a rhombus (Figure 4).

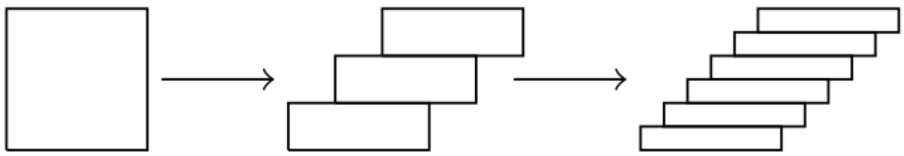

Fig. 4: *A visual demonstration of Cavalieri's principle in the case of a square.*

As these pieces get smaller and smaller, the shape starts to resemble a rhombus more and more closely while its area remains the same. Identifications measures are those actions taken by a teacher in an effort to identify any misconceptions that students may have about a concept. These measures consist of a variety of formative assessment practices that seek to penetrate to the origin of student errors. For this reason, these sorts of assessments should seek to ascertain as much information as possible from students so that their exact mistakes may be identified. Consider the following question:

Is there a triangle with sides of length 3, 4, and 8 inches?

Giving this question to students is likely to result in simple yes or no responses. We know that the correct response in this case is no, which follows from the triangle inequality, but a student who responds correctly may not do so for the right reasons. Perhaps the student simply tried to sketch out a triangle with these sides, but found that they were unable to. If this is the case, we would find that this student may not answer correctly if we change the lengths to 100, 120, and 300 feet. Additionally, with straightforward alternative response items such as this, there is always the chance that the student simply guessed. On the other hand, a student who replies "yes" might be very familiar with the triangle inequality, but happened to simply misread the question. In this situation, asking students to justify their responses can help to remove the ambiguity about their answers and allow the teacher to identify and address any misconceptions that might exist.

Once misconceptions are identified, teachers can correct student misconceptions through a multitude of measures, which include activities such as asking responsive follow up questions for clarification, reteaching, working with individual or groups of students, engaging the class in discussion, and so on. It is important to note that the teacher should do their best to impart a correct understanding of the concept before any misconceptions are considered. Let's say that a number of students made the following mistake during the identification phase:

$$3(x + 7) = 3x + 7$$

that is, they distributed the 3 to the first term, but not to the second term. It would be simple for the teacher to simply reiterate that the 3 must be distributed to both terms in the quantity, but evidence surrounding this mistake suggests that students do not understand the process of distributing, potentially regarding it as an arbitrary mathematical rule. If the teacher only provides them with procedural-driven feedback on the way to distribute through a "rule," it still remains nothing but an arbitrary rule, and this teacher can only hope that students do not forget this rule. Alternatively, It would be better to demonstrate the calculation in a way that allows for a greater conceptual understanding of the distributing:

$$\begin{aligned} 3(x+7) &= (x+7)+(x+7)+(x+7) \\ &= x+x+x+7+7+7 \\ &= 3x+37 \\ &= 3x+21 \end{aligned}$$

Here instead of a rule, students are given a process that can be understood and applied to new scenarios, for example, the expression $3(x+y+7)$. The arbitrary "rule" now appears as nothing more than a deep link for a well known process. We note that fostering these kinds of deep connections is emphasized by NCTM's Process Standards and Teaching and Learning Practices (NCTM, 2008; NCTM, 2014).

Of these three measures, prevention measures generally come more easily to veteran teachers since they have had a longer opportunity to acquire a prior knowledge of common student misconceptions of the concept. Much of this knowledge comes naturally from teaching experience, so, in general, prevention measures taken by experienced teachers may be more effective than those taken by their relatively inexperienced colleagues. Inexperienced teachers must rely on their own experience learning the topic, their intuition as to what might confuse the students, as well as outside information from literature or from the advice of fellow teachers. Despite the challenges associated with anticipating, addressing, and preventing misconceptions, it is an action which should be prioritized by all teachers so that students can develop a stronger and more accurate understanding of mathematical phenomena, thereby reducing the time necessary for identifying and correcting misconceptions.

3 Preventing Student Misconceptions

In the case of tangent lines, the prevention of misconceptions must begin by providing students with a comprehensive definition of the concept. Without a proper definition of what a tangent line actually is, it is challenging for students to properly identify and construct tangents. For a topic as difficult for students as tangent lines, it may be tempting to start out by providing a non-rigorous and 'intuitive' definition, as many textbooks do, such as "a tangent line is a line that barely touches the function," "a tangent line hugs the function at the point of tangency," or "a tangent line 'just touches' a curve" (Kajander & Lovric, 2009). However, these pseudo-definitions can often create confusion. What does it mean for a line to "barely touch" a function? Do a pair of intersecting lines barely touch each other, or, further, does not any line that intersects a function at a single point "barely touch" that function? So, a student may wonder if a tangent line is simply a line that intersects a function at a single point. Under the standard calculus definition of tangency this is clearly not the case, yet these are the types of ideas that can take root if we use this sort of "intuitive" language in an attempt to ease students into the topic.

Students can, of course, develop misconceptions about tangent lines even if we are careful with our use of language when introducing the concept. Starting with a proper definition of tangency, however, gives us a strong foundation on which to address these misconceptions and construct proper concept images, as demonstrated below.

Before we discuss further prevention measures for misconceptions on tangent lines, we first must clarify exactly what misconceptions we are aiming to prevent. It is, of course, nearly impossible to entirely address all potential misconceptions, however, we should do our best to cast a wide net. We will concern ourselves here with two properties held by some specific tangent lines which are often mistakenly taken as necessary conditions of tangency in general, namely the property that a given tangent line only intersects a function at a single point (property 1) and the property that a given tangent line does not cross over a function (property 2), as well as the local versions of these ideas.

Biza et al. (2008) have shown how students may take properties 1 and 2 as necessary (and possibly sufficient) conditions of tangency in their current form or in an altered local form (Biza, Christou & Zachariades, 2008). Property 1, for example, may be transformed into a local property by a student by stating that in some neighborhood surrounding the point of tangency, a tangent line intersects a function at a single point (Biza, Christou & Zachariades, 2008). This is similar, though not identical to, the statement that a tangent line cannot coincide with the function around the point of tangency, which is another misconception students might develop. Property 2 is commonly made into a local property by stating that a tangent line cannot cross over a function at the point of tangency

We have outlined above some common misconceptions that students may develop about tangent lines in calculus, now we will show how easily these misconceptions can be disproven once we have a proper definition of tangency. As Winicki-Landman and Leikan note, the only correct definition of tangency, i.e., the only definition that provides a necessary and sufficient condition of tangency, is that the tangent line is the limiting position of secant lines passing through the point of tangency (Winicki-Landman & Leikin, 2000). Our examples in this paper will all be differentiable functions, hence the tangents to these functions are those lines passing through the point of tangency which have slopes equal to the value of the function's derivative at this point. The misconceptions concerning property 1 can all be dealt with by a single example, namely the tangent line to a non-vertical line at any point (Figure 5).

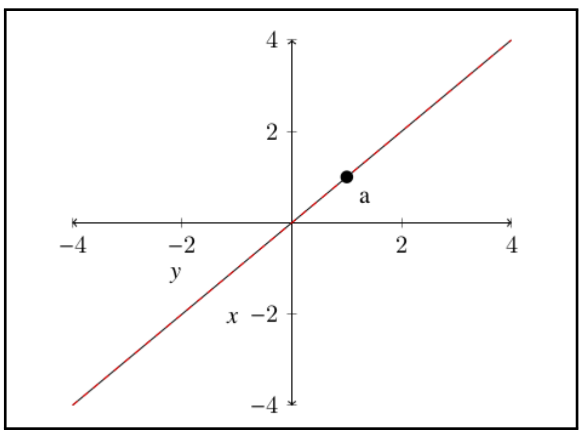

Fig. 5: *The tangent line to any linear function coincides with that linear function.*

It is clear that, given the definition above, this tangent is none other than the line itself, meaning that not only does the tangent coincide with our function in a region around the point of tangency, it in fact coincides with the function everywhere. It is clear that the other misconceptions concerning property 1 are thus disproved. A single example will also suffice to disprove the misconceptions concerning property 2, shown below (Figure 6). The line in Figure 6, which is easily shown to be a tangent under our definition, crosses over the function at the point of tangency, thus disproving the misconceptions concerning property 2.

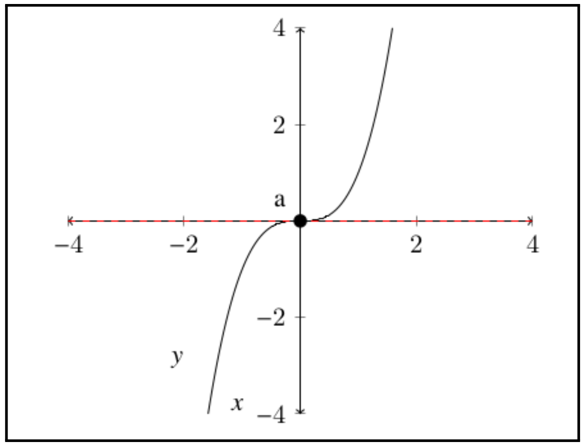

Fig. 6: *Students were asked to draw the tangent to the function at point a or explain why it does not exist.*

It is evident from the examples above that we can easily show students why certain misconceptions are incorrect; all that is required is a sound counter-example. Yet, such an approach does nothing but render the learning experience into a passive one. In this case, all we can do is hope that the students have received the information properly. It is much more desirable to engage students in the active process of thinking about these misconceptions on their own or in a small group. Fortunately, this topic lends its self to a constructive process of discovery. To this end, a teacher might ask the students to consider whether the following statements are true, and to give their reasoning (a counter-example in the case of false statements):

1. A tangent line can only intersect a function at a single point.
2. There is a region around the point of tangency in which the tangent line only intersects the function at a single point.
3. A function can have the same line as a tangent to two separate points.
4. A tangent line cannot coincide with the function in a region around the point of tangency.
5. A function can be its own tangent line.
6. A tangent line cannot cross over the function.
7. A tangent line cannot cross over the function at the point of tangency.

While these are several examples of the types of questions that may be asked, it should be noted that they are all alike in that they probe the students to explore specific ideas about tangent lines.

4 Identifying Student Misconceptions

Despite our best efforts to prevent misconceptions in a lesson, very rarely can we expect an entire class to achieve that goal. It is then necessary to identify both those students who have developed misconceptions and the nature of the misconceptions that they have developed. This is typically done through formative assessment. Formative assessment has a dual benefit: it allows students to potentially recognize their own misconceptions, and it forces them to record their own misconceptions in their work, which can then be analyzed by the teacher.

In creating a formative assessment for the identification of student misconceptions, teachers should do their best to avoid students 'slipping through the cracks.' This means that the teacher should do their best to draw out as many student misconceptions as possible, within reason. In most situations, the best way to do this is by providing a wide range of problems concerning the concept. In the case of tangent lines, for example, such a formative assessment might provide students with graphical examples that ask students to draw a tangent line if it exists, or to identify whether a given line is, or is not, a tangent line. The authors had great success drawing out student misconceptions with these sorts of questions in a previous study. Some examples from that study are given below (Figures 7 and 8).

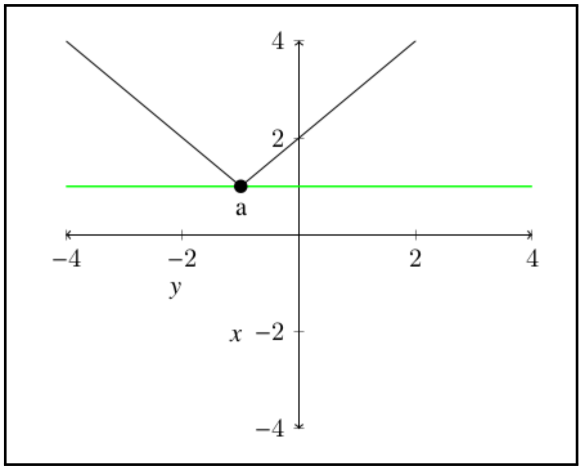

Fig. 7: *Students were asked to explain why the green line is or is not a tangent.*

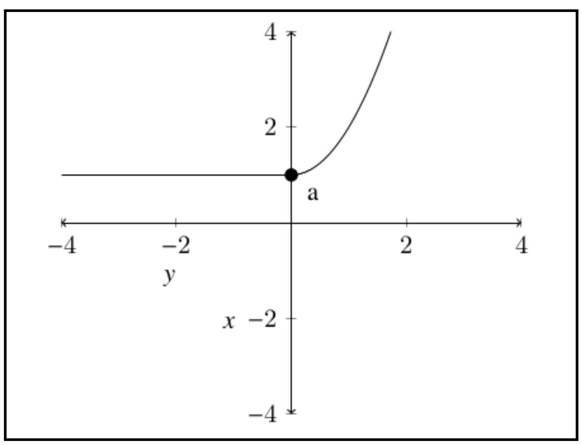

Fig. 8: *Students were asked to draw the tangent to the function at point a or explain why it does not exist.*

Identification does not stop once the students have completed the assessment; the results, whether verbal or written, must be analyzed by the teacher so that they are able to determine the actual misconceptions held by various students and, to a greater or lesser extent, the source of these misconceptions. It is important to note that the goal of formative assessment should not be to make students recognize their own misconceptions, although this is not an undesirable result, but to allow the teacher to identify these misconceptions so that they can be addressed in a proper manner. When a student identifies a personal misconception, they are really discovering a contradiction between their own concept and some aspect of the concept itself; that is, they are discovering a potential conflict factor. The student may very well resolve this contradiction with a change to their concept image, but there is no guarantee that this new concept image will no longer be in contradiction with the actual concept.

An example involving a participant (Student B) in the authors' previously mentioned study will help to make this problem clearer:

> Student B was initially very firm in their assertion that a tangent line could not cross over a function at any point. To explore this idea further, we noted that Student B had correctly identified the tangent line at the minimal point of a parabola, then proceeded to ask her if by changing the function to the right of this minimal point whether the line would cease to be tangent to the function. Student B responded that the line in the new graph would still be tangent to the function, hence concluding that a tangent line could in fact cross over a function. Yet, Student B was unable to accept tangent lines that passed over the function at the point of tangency. In the end, Student B defined a tangent line as a line which intersects a function without crossing over the function at the point of tangency.

Figures 9 and 10 are similar to the graphs mentioned in the above quote.

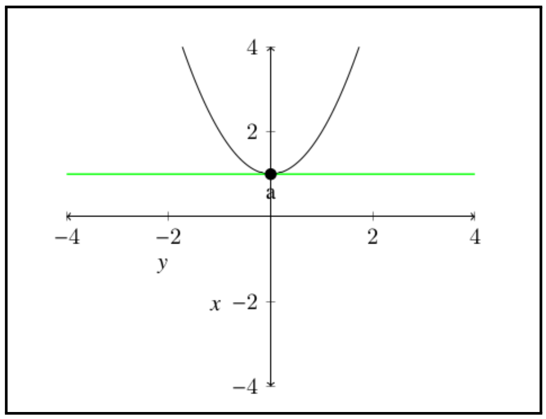

Fig. 9: *Graph of parabola with tangent shown at the minimal point.*

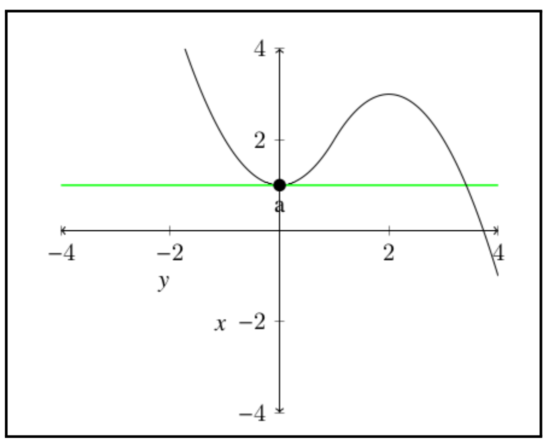

Fig. 10: *New graph altered to the right of the minimal point.*

In the previous example, Student B begins with a personal concept definition that does not allow for tangent lines to cross over functions. After an encounter with an actual example of the concept of the tangent line (Figure 10), she changed this definition so that the contradiction between her personal definition and this actual example of the concept is remedied. Indeed, Student B's new definition (that a line is a tangent if and only if it intersects the function and does not cross over the function at the point of tangency) allows for a correct identification of the tangent lines in Figures 9 and 10, albeit for the wrong reasons. We have already shown (see Figure 6) that this new conception of tangency is still in contradiction with the actual concept of tangency. As a direct example of the inadequacy of this new definition, it allowed Student B to accept multiple tangent lines at a given point, which they did in the case of a parabola (Figure 11).

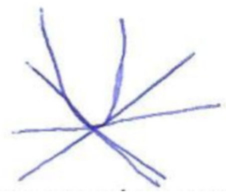

Fig. 11: *Student B accepted the following lines as all being tangent to the minimal point of a parabola. It is important to note that she thought that these lines did not cross over the parabola at the minimal point, when in reality this would only be true for the 'horizontal' line.*

5 Correcting Student Misconceptions

The final stage in the treatment of student misconceptions, correction, will typically vary from teacher to teacher, but there are some general principles that should be observed. Treatment of misconceptions can be subdivided into positive and negative actions. Positive actions are those actions which are intended to construct correct conceptions of a topic, e.g., providing a correct definition, showing proper examples of a concept, solving problems, etc. On the other hand, negative actions are intended to deconstruct improper and incorrect portions of the students' concept images. Examples of negative actions include providing counterexamples to certain false statements, showing why certain definitions of a concept are incorrect, and providing false examples of a concept and explaining why they are incorrect. As a general principle, positive actions should precede negative actions, as the 'correct' ideas constructed through positive actions should be used to deconstruct the 'incorrect' ideas through negative actions. For example, if we consider a student who believes that tangent lines can only intersect a function at a single point, we are unable to provide a proper counterexample to this statement until we provide the student with the correct definition of a tangent line. Only when this definition has been understood by the student does constructing a counterexample to the incorrect statement become possible.

Teachers should use the information gained from the identification stage to guide their actions during the correction stage. Emphasis should be placed on those misconceptions that students exhibited in the prior formative assessment. On the other hand, less time, if any at all, should be spent on correcting misconceptions that were not present in a given class.

6 Conclusion

As we have seen, misconceptions are not an issue which can simply be solved by trivial corrective actions, but must be dealt with through a definite and comprehensive plan. Such a plan should always start with introducing students to the formal definition of the concept in question. Without this formal definition, we have no basis on which to construct examples and counterexamples that demonstrate the various properties of the object of study. Furthermore, a lack of a formal definition can cause problems for students as they will find themselves unable to test their own conception of the concept against the formal theory. From here, teachers can begin to take the necessary actions to prevent, identify, and correct student misconceptions. Using the approach we have described here will help students to acquire am accurate and more complete understanding of a given concept.

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Mark D. Hogue is an assistant professor in the Secondary Education Foundations of Education Department at Slippery Rock University of Pennsylvania. Previously, Dr. Hogue was a high school mathematics teacher and high school principal. In addition to teaching middle level and secondary mathematics methods, quantitative research methods, and assessment courses, Dr. Hogue is also the graduate coordinator for the principal certification program at SRU. In addition to conducting research in mathematics education, he serves as the evaluator for several grants.

Dominic Scarcelli received his B.S. in mathematics from the University of Pittsburgh in 2019 and is currently a graduate student in the College of Education at Slippery Rock University in Slippery Rock, Pennsylvania. Dominic comes from a long line of Scarcelli teachers, as his father, grandfather, and great-grandfather had also been mathematics educators. Upon graduating from Slippery Rock, Mr. Scarcelli looks forward to teaching secondary level mathematics students in area schools so that he can share his passion for mathematical problem solving with young learners.