Less is More: Right Triangle Relationships

Lisa Jacob, *Mason High School*

Abstract: As educators, we must decide which tools are the most strategic to use when designing and implementing classroom lessons. With numerous free dynamic softwares and pricey handheld tools available, it can become overwhelming to create and administer daily lessons to our students. After reflecting on a two-day special right triangle lesson I facilitated, I learned that "less is more."

Keywords: geometry, Pythagorean theorem, right triangles

1 Introduction

The formula $a^2 + b^2 = c^2$ is one that most tenth grade geometry students have memorized; however, the Pythagorean theorem is not always the most efficient formula to use when finding a missing side measurement in a right triangle. With special cases of right triangles, relationships can be applied to determine the length of a missing side, instead of invoking the Pythagorean theorem to isolate a missing variable. I facilitated a two-day lesson on special right triangles in a regular geometry classroom. My students explored isosceles right triangle relationships through a paper and pencil activity and investigated $30°$-$60°$-$90°$ triangle relationships with Desmos.

After sharing my original lesson plan and handouts with a group of teachers in a graduate-level advanced teaching methods course, I learned that "less is more." As educators, we teach students how to use appropriate tools strategically when presented with different problems to solve; as educators, we too must decide which tools are the most strategic to use when designing and implementing classroom lessons (Ohio Department of Education, 2017). After reflecting on student data, reviewing articles, and discussing the lesson with fellow educators, I decided that facilitating paper-and-pencil activities can be more strategic than facilitating computer-based activities when exploring the relationships in special right triangles.

2 Demographics

I've taught mathematics at William Mason High School (MHS) for the past two years. The classroom from which I gathered evidence was my fourth bell— a regular geometry class comprised of 26 students (14 females and 12 males; 2 ninth graders and 24 tenth graders). Seven of the students were taking two math courses simultaneously—namely, Geometry and Algebra II. One student had a one-on-one aide to assist him with an anger disorder. I had an additional aide to help with required accommodations and modifications for five of the learners.

In this classroom, there is a large achievement gap. Some learners arrive to class with mastery levels of understanding while other students need to practice a newly taught skill repetitively before

feeling confident. To accommodate the wide variety of learners at MHS, the district has a "focus on making learning more personalized, where educators recognize that we must empower students to learn about themselves so that they can take more ownership over the learning path they choose" (Mason City Schools, 2018). By teaching students a variety of skills, we, as educators, are planting the seeds for learners to further explore their own passions.

3 Summary of the Original Lesson

At the conclusion of the first day, students should be able to explain how to determine the length of the hypotenuse of a 45°-45°-90° triangle if they know the length of the legs and how to find the length of the legs of a 45°-45°-90° triangle if they know the length of the hypotenuse. At the conclusion of the second day, students should be able to explain how to find the lengths of the long leg and the hypotenuse of a 30°-60°-90° triangle, if the short leg is n units long.

As shown in Figure 1, a student sketched an isosceles right triangle with leg lengths of three units and used the Pythagorean theorem to solve for the hypotenuse length. The student was able to express the hypotenuse length in simplest radical form.
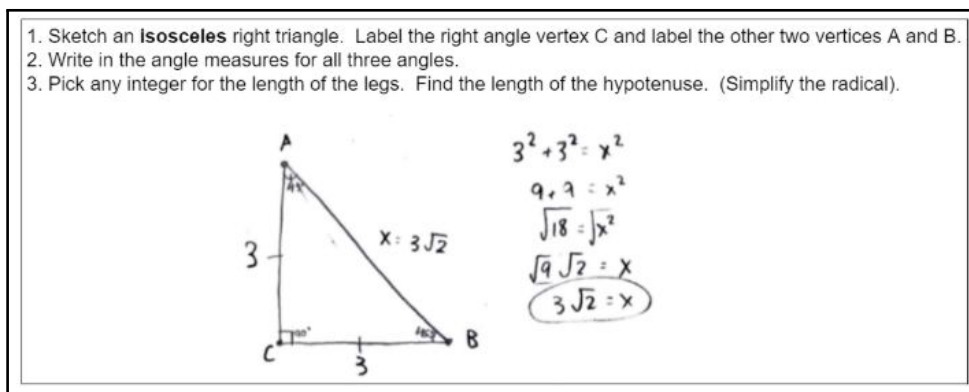

1. Sketch an **isosceles** right triangle. Label the right angle vertex C and label the other two vertices A and B.
2. Write in the angle measures for all three angles.
3. Pick any integer for the length of the legs. Find the length of the hypotenuse. (Simplify the radical).

$$3^2 + 3^2 = x^2$$
$$9 + 9 = x^2$$
$$\sqrt{18} = \sqrt{x^2}$$
$$\sqrt{9}\sqrt{2} = x$$
$$3\sqrt{2} = x$$

Fig. 1: *How a student solved for the hypotenuse length in an isosceles right triangle with integer leg lengths.*

After sharing data with their peers, students made observations about the relationship between the length of the legs (when the legs are integer values) and the length of the hypotenuse of a 45°-45°-90° triangle. Figure 2 shows how one student generalized the relationship between the leg lengths and the hypotenuse length in a 45°-45°-90° triangle.
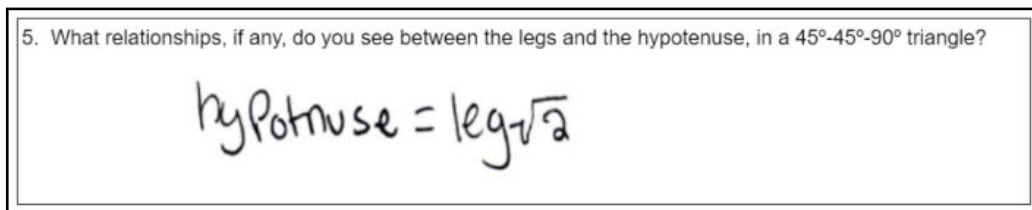

5. What relationships, if any, do you see between the legs and the hypotenuse, in a 45°-45°-90° triangle?

hyPotnuse = leg√2

Fig. 2: *How one student generalized a relationship in all isosceles right triangles.*

Next, students were instructed to draw another isosceles right triangle. This time, the hypotenuse of the isosceles right triangle had to be an integer. Figure 3 shows how a student used the Pythagorean theorem to find the lengths of the legs of an isosceles right triangle with a hypotenuse length of five units. As a final step, the student rationalized the length of the legs in their triangle.
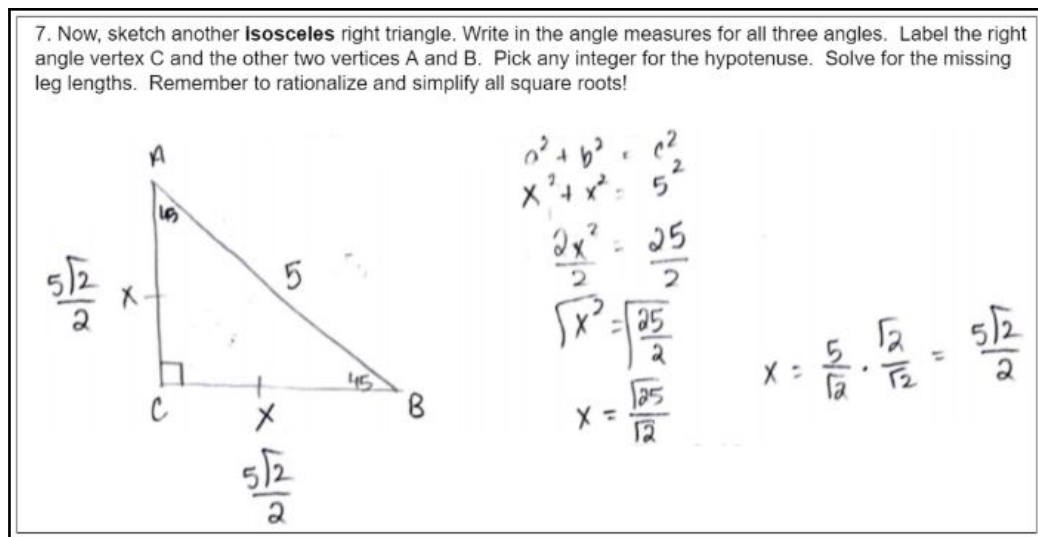
Fig. 3: *How a student solved for the leg lengths in an isosceles right triangle with an integer hypotenuse.*

Figure 4 shows how a student collected data from their peers to analyze another pattern in an isosceles right triangle; the student analyzed how the leg lengths of an isosceles right triangle can be found when the hypotenuse is an integer measure.
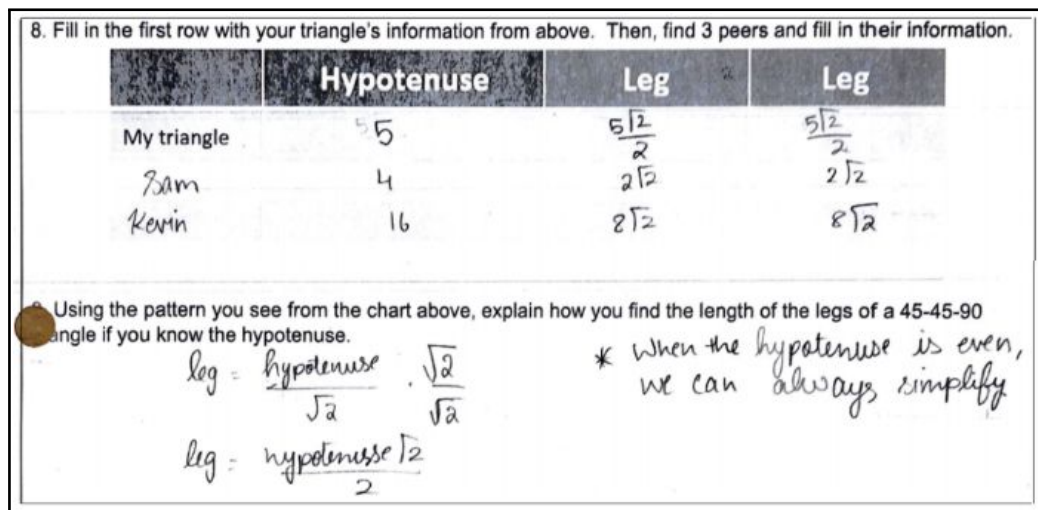

Fig. 4: *A student collected data to generalize another relationship in all isosceles right triangles.*

During the second day of the lesson, students analyzed the relationships in $30°$-$60°$-$90°$ triangles by first analyzing an equilateral triangle. Figure 5 shows how a $30°$-$60°$-$90°$ triangle is created by making AB the perpendicular bisector of CD in $\triangle BCD$. $\triangle ABC$ is an equilateral triangle, where the length of CA is half the length of BC, since A is the midpoint of CD. Through repetition, students observed that the length of the hypotenuse is double the length of the short leg and that the length of the long leg is the length of the short leg multiplied by $\sqrt{3}$.
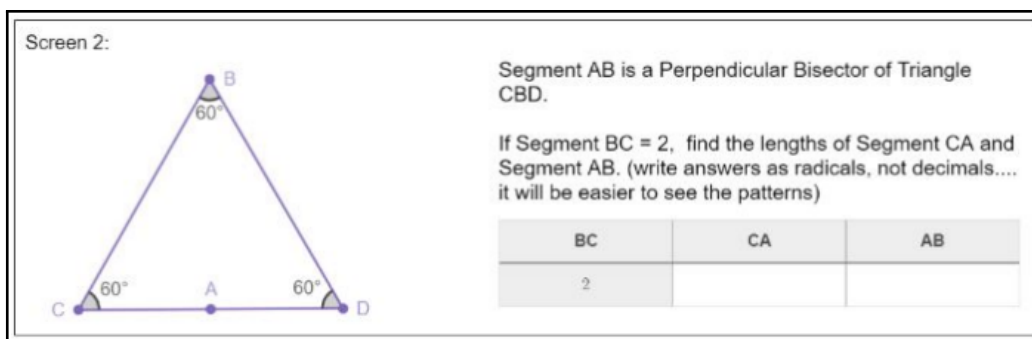
Fig. 5: *A screenshot of Screen 2 from the Desmos activity students completed on their personal devices.*

4 Student Data

Throughout the course of the two-day lesson, students were given time to work at their own pace with their partner (or individually if they preferred) and then were brought together at various checkpoints to discuss big ideas as an entire class. As students were working through the investigation, I would engage them in a conversation when I noticed an error. Without telling them they were incorrect, I would ask them to explain their problem-solving strategy. As students verbally summarized their problem-solving strategies to me, they would realize they made a mistake and they would rework the problem. When I collected the handouts from my students after the two-day lesson was completed, I noticed that all my students had erased their errors and recorded only correct work. Through video evidence, I was able to record and quantify common student errors.

In my classroom of 26 learners, I observed four students using the Pythagorean theorem incorrectly. Figure 6 showcases how a student used the formula $a^2 \cdot b^2 = c^2$ to determine the length of the hypotenuse instead of $a^2 + b^2 = c^2$.
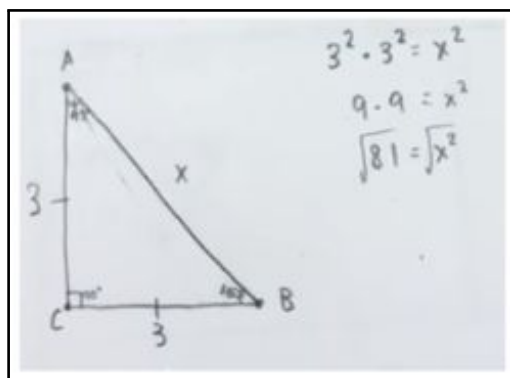

Fig. 6: *A student's misapplication of the Pythagorean theorem.*

When I asked this student how they determined the length of the hypotenuse, they said they used the Pythagorean theorem; when I asked them what the Pythagorean theorem was, they immediately realized they had used a multiplication sign instead of an addition sign on their paper.

I observed three students make errors when filling in a table of side length measurements; after considering the first row of given data in the table, some students repeated an inaccurate relationship. In Figure 7, I recorded a student explaining that the relationship between AC (the short leg) and CB (the long leg) is "to add the short leg (of one unit) to the hypotenuse and square root it

to find the other leg." When I asked the student to prove this relationship through the Pythagorean theorem with their data of $AC = 2$, $AB = 4$ and $CB = \sqrt{5}$, they realized that this relationship was not accurate. Through the Pythagorean theorem, they eventually concluded that the long leg is the short leg multiplied by $\sqrt{3}$.
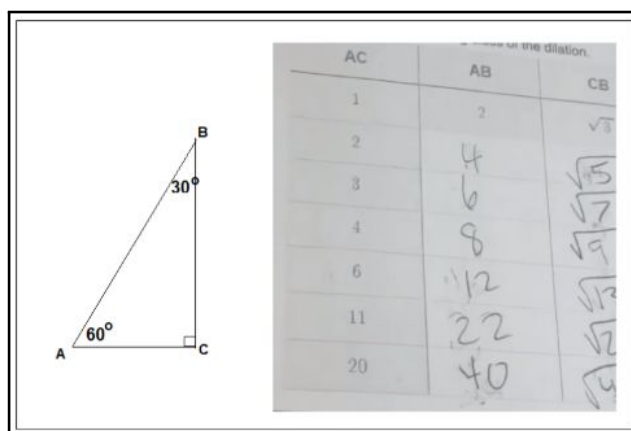

Fig. 7: *A student incorrectly completed the chart when determining the length of CB.*

I recorded three students who heavily relied on their calculators to find the lengths of missing sides in special right triangles; all three of these students were unable to report missing side lengths of special right triangles in simplified radical form. Figure 8 shows how a student used the Pythagorean theorem to find the lengths of the missing legs in the isosceles right triangle. When solving algebraically for x, the student "got stuck" when they realized their calculator would not report $\sqrt{112.5}$ as a simplified radical (See Figure 9).
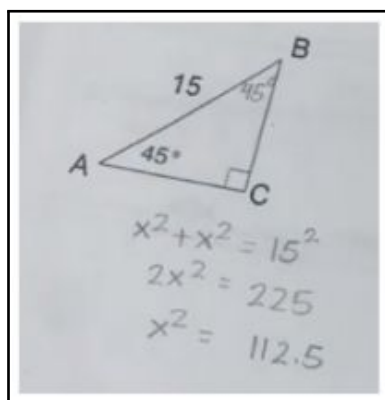

Fig. 8: *How one student used the Pythagorean theorem to solve for the missing leg lengths in a 45°-45°-90° triangle with a hypotenuse length of 15 units.*
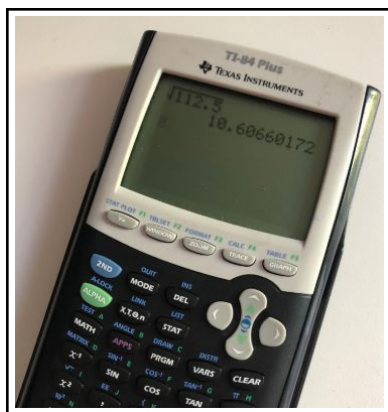
Fig. 9: *One student "got stuck" when their calculator would not report $\sqrt{112.5}$ as a simplified radical.*

5 Review of the Literature

Jennifer Wilson is a high school teacher in Mississippi who shared a special right triangle lesson she facilitated in a blog post in 2015 (See Wilson's original post here: `https://tinyurl.com/sw7sman`). Wilson (2015) had her students explore relationships of a 45°-45°-90° triangle by looking at half of a square and 30°-60°-90° triangle relationships by looking at half of an equilateral triangle. Through TI-Nspire software, students moved vertices of a triangle to analyze patterns in data. Wilson chose to "not tell" her students the relationships between the legs and the hypotenuse in a 45°-45°-90° triangle and the short leg to the long leg and the short leg to the hypotenuse in a 30°-60°-90° triangle. One of Wilson's students shared, "I found out that when you memorize something, you'll eventually forget it, but if you try to learn it and understand it, the info just sticks to you."

Kurz and Lee (2018) write about how the use of tools can provide a deeper level of understanding for learners. Specifically, they focus on how the use of AngLegs can help students conceptualize patterns and relationships in right triangles. Kurz and Lee (2018) state, "AngLegs are straight plastic tools that can be used to create polygons by easily connecting and disconnecting plastic pieces to test and retest conjectures. These tools are more effective than pencil-and-paper explorations in that they are durable, hands-on tools; they provide visual support so learners can easily measure angles or compare features without tedious constructions or calculations" (p. 226). Figure 10 shows how at least four different isosceles right triangles can be created and measured using AngLegs.
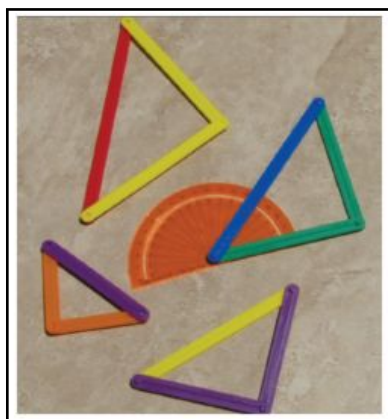

Fig. 10: *Four different isosceles right triangles made with AngLegs.*

6 Revision Analysis

After reviewing literature and sharing this lesson with 12 math educators in a graduate course I am taking, I have made numerous revisions to my lesson. My revised lesson plan (See Appendix A) and revised student handouts (See Appendix B) showcase how overall, I have shortened the two-day investigation under the pretense that "less is more." In the revised student handout, I have eliminated many of the repetitive practice problems (see Figure 11). After reflecting over conversations I had with my peers, it was a unanimous consensus that practicing a skill over a prolonged period of time will commit the strategy to memory as opposed to practicing a skill repetitively in an isolated environment.
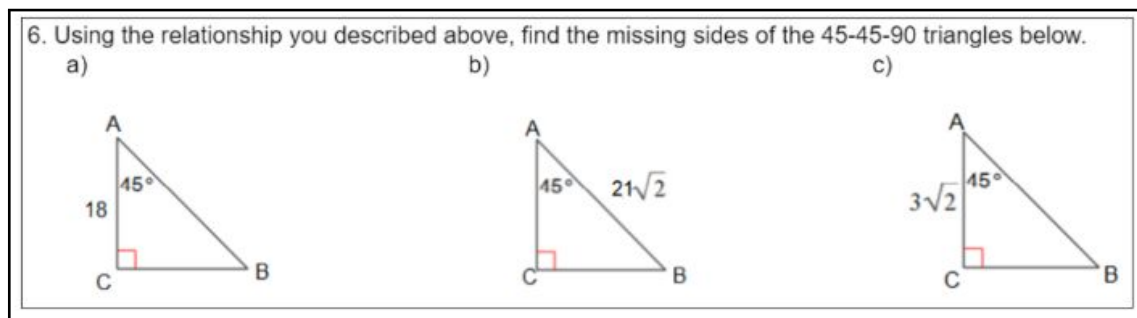

Fig. 11: *A set of practice problems that were eliminated when creating the revised handout.*

A drastic revision was eliminating the Desmos component for the 30°-60°-90° triangle investigation. I noticed that when learners were using Desmos as a dynamic software to explore patterns in 30°-60°-90° triangles, they weren't conversing with their peers. While engaging with the software, students were so focused on watching the summary screen (see Figure 12) to receive feedback that they stopped sharing problem-solving strategies with their peers. By revising the student handout to a paper-and-pencil format, I anticipate student collaboration to increase.

Fig. 12: *The anonymized Desmos summary page provides immediate feedback to students when projected. A "check" reflects a correct response and an "x" reflects an incorrect response.*

Since the Desmos activity was eliminated in my revised lesson, and not having AngLegs in my classroom, my professor created a free dynamic applet to assist students while completing their handout. By using the GeoGebra Right Triangle Explorer Applet (See applet here: `https://tinyurl.com/vwezsed`), students are able to view and analyze many 45°-45°-90° triangles and 30°-60°-90° triangles in a short amount of time (See Figure 13).
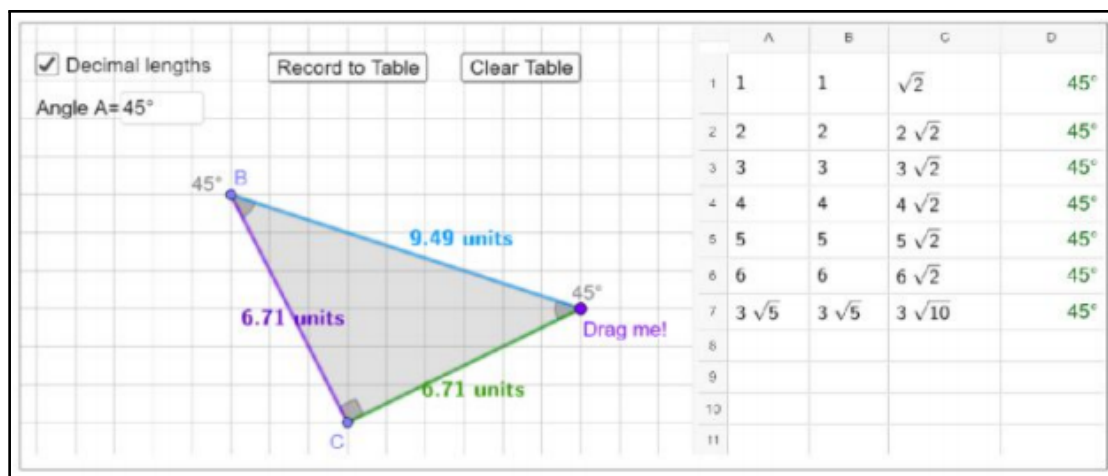
Fig. 13: *The Right Triangle Explorer GeoGebra Applet records all side measures of the triangle in exact form in the table and approximates the side lengths of the triangle in the diagram.*

7 Conclusion

It is not often in everyday life that someone finds themselves solving for a missing side length in a right triangle. It can be argued that many concepts and skills taught in elementary and secondary schools are not applicable to everyday life; however, as individuals, we are always tasked with problems to solve in our day-to-day lives. From creating a schedule of daily to-dos, to creating a weekly budget, to finding a quickest route from one place to another, we are strategically using problem-solving techniques to solve our dilemmas efficiently.

As an educator, I often refer to our minds as toolboxes and I refer to problem solving strategies as our tools. In today's world, where time is equivalent to money, we should be working smarter and not harder; we should be using "tools" that will help us find a solution the quickest. By allowing students to analyze patterns in special right triangles, they can save time by using the patterns in Figure 14 instead of using the Pythagorean theorem.
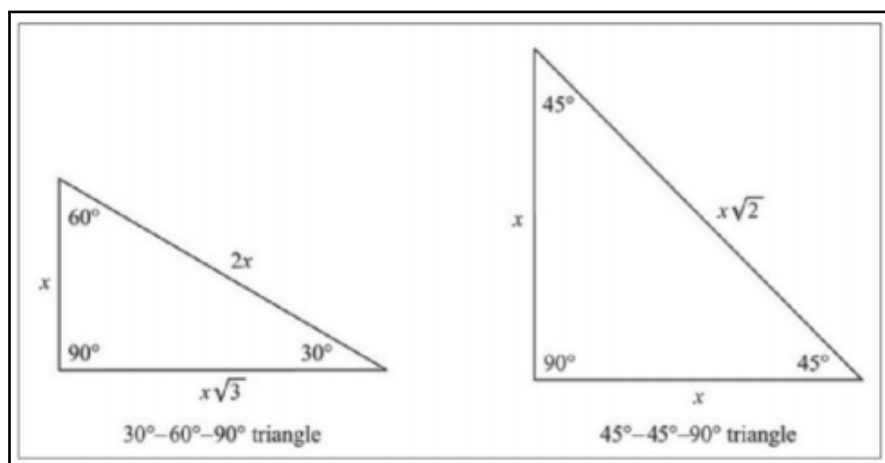

Fig. 14: *Special right triangle relationships that are commonly presented to students in textbooks.*

By choosing the most efficient problem-solving strategy, we adopt the understanding that "less is more."

References

Kurz, T. & Lee, Y. (2018). Using tools to make sense of right triangles. *National Council of Teachers of Mathematics, 13*(4), 226-230.

Mason City Schools. (2018, November 26). Focus on Big Rock 3: Personalized learning. Retrieved from `http://www.masonohioschools.com/cms/one.aspx?portalId=391119&pageId=16890261`

Ohio Department of Education. (2017). Ohio's learning standards for mathematics: Standards for mathematical practice. Retrieved from `http://education.ohio.gov/Topics/Learning-in-Ohio/Mathematics/Ohio-s-Learning-Standards-in-Mathematics`

Public School Review. (2003-2019). *William Mason High School*. Retrieved May 2019, from `https://www.publicschoolreview.com/william-mason-high-school-profile#reviews`

U.S. News & World Report. (2019). *William Mason High School*. Retrieved May 2019, from `https://www.usnews.com/education/best-high-schools/ohio/districts/mason-city-school-district/william-mason-high-school-15778`

Wilson, J. (2015, June 22). Ask, don't tell (part 3): Special right triangles. Retrieved from `https://www.nctm.org/Publications/Mathematics-Teacher/Blog/Ask,-Don_t-Tell-(Part-3)_-Special-Right-Triangles/`

Lisa Jacob, `suberllm@miamioh.edu`, is a mathematics teacher at Mason High School in Mason, Ohio. She is a graduate of Miami University, and is currently in her 12th year of teaching.

APPENDIX A: Revised Lesson Plan

Lisa Jacob
EDT 566 Lesson Plan (2019)

Lesson Title
Special Right Triangles Investigation (*Adapted from MCTM, June 2015*)

Grade Level
10th Grade (Regular Geometry)

Class Background
The video and picture evidence I captured of learner work is from a regular geometry class at Mason High School. The class is comprised of 26 students, where a large achievement gap is present. Two students are Freshman and 24 students are 10th graders. Seven of the students in this class are taking two math courses simultaneously throughout the semester (geometry and algebra II). One of the students in the classroom has a 1-on-1 aide to assist them with their anger disorder. I have an additional aide in this classroom to help with required accommodations and modifications for five of the learners.

Lesson Objectives
This two-day lesson provides learners the opportunity to explore the relationships in 45°-45°-90° and 30°-60°-90° triangles. By use of the Pythagorean theorem, learners will discover relationships in isosceles right triangles and equilateral triangles, when the amplitude is constructed. At the conclusion of this two-part investigation, learners will be able to summarize the relationship between a leg length and a hypotenuse length in an isosceles right triangle and the relationships between the long leg and the short leg, and the hypotenuse and short leg in any 30°-60°-90° triangle.

This lesson is facilitated during the last Unit of study: Polyhedrons. In this Unit, students use concepts learned throughout the course to construct a Polyhedron of their choice. Students will use knowledge of special right triangles, trigonometry, surface area, volume and construction techniques to complete this final summative assessment.

Ohio's Learning Standards

CRITICAL AREA OF FOCUS #3
Similarity, Proof, and Trigonometry
Students apply their earlier experience with dilations and proportional reasoning to build a formal understanding of similarity. They identify criteria for similarity of triangles, use it as a familiar foundation for the development of informal and formal proofs, problem solving and applications to similarity in right triangles. This will assist in the further development of right triangle trigonometry, with particular attention to special right triangles, right triangles with one side and one acute angle given and the Pythagorean Theorem. Students apply geometric concepts to solve real-world, design and modeling problems.

G.SRT.8 Solve problems involving right triangles.★ a. Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems if one of the two acute angles and a side length is given. (G, M2)

Materials, Technology, Resources

Student Resources:

Students will all be given the Special Right Triangles Investigation Handout. Students should each have a personal computer (with internet accessibility) to interact with the Right Triangle Explorer during the two-day lesson. A computer is also required for students to complete the Desmos Homework assignment, at the completion of the two-day lesson. Although it is not required, a calculator could be used to verify computations throughout this activity.

Teacher Resources:

Teachers will need to project the Warm Up PowerPoint to lead students through the pre-required skill of simplifying radicals and rationalizing ratios. Although it is not required, a Wacom Tablet and Smart Notebook software is useful to model problems to the class (The modeled problems can be saved as a PDF and electronically shared with students to review throughout the Unit).

Lesson Procedures Day 1 (Warm Up, Lesson Core, Closure, Homework)

Warm Up (15 minutes)

1. To begin this lesson, the instructor will facilitate the Warm Up Activity.
2. The instructor will circulate around the classroom to ensure partnerships are sharing ideas; the instructor will challenge partnerships who are sharing misconceptions.
3. The instructor will model learner work via the Wacom Tablet and Smart Notebook software to the class; the Tablet notes will be shared with all learners to reference throughout the Unit.

Lesson Core (30 minutes)

Following the Warm Up Activity, the instructor will allow partnerships to work at their own pace through Investigation 1. As partnerships are progressing through the activity, the teacher will be circulating to each partnership to check for accuracy. The teacher will have partnerships summarize their work when accurate and will challenge solutions when there are misapplications.

Closure of Day 1 (5 minutes)

During the last 5 minutes of class, the teacher will ask learners to share their responses to Question 5, in the Investigation. The teacher will capture learner responses on the Wacom Tablet to verify that all partnerships were discovering the same relationships in a 45°-45°-90° triangle. The teacher will ensure that all learners understand that the hypotenuse is the length of the short leg multiplied by $\sqrt{2}$.

Homework

For homework, learners will be expected to complete the "45°-45°-90° Practice" problems.

Lesson Procedures Day 2 (Review, Lesson Core, Closure, Homework)

Review (10 minutes)

1. To begin this lesson, the instructor will have learners share their solutions to the "45°-45°-90° Practice" homework problems. The teacher will record the learner's problem-solving routine on the Wacom Tablet, using the Smart Notebook software.
2. During this time, learners can revise mistakes and ask questions on any misconceptions.

Lesson Core (35 minutes)
Following the Review Activity, the instructor will distribute the "Day 2" handout to all students. The instructor will allow partnerships to work at their own pace through questions #1-6. As partnerships are progressing through the activity, the teacher will be circulating to each partnership to check for accuracy. The teacher will have partnerships summarize their work when accurate and will challenge solutions when there are misapplications.
  ● If the instructor finds an error in a student's table, the instructor will ask the student to use the (incorrect) side lengths to show how the Pythagorean theorem applies to the right triangle. Through the Pythagorean theorem work, students will realize they have made a mistake; they will then work on correcting their error.

Closure of Day 2 (5 minutes)
During the last five minutes of class, the teacher will ask students to share their responses to question #6. The teacher will capture students' responses on the Wacom Tablet to verify that all partnerships were applying the correct relationships in a 30°-60°-90° triangle.
  ● To generalize the relationships in a 30°-60°-90° triangle, students should explain that if the short leg is n units, the long leg is $n\sqrt{3}$ units and the hypotenuse is $2n$ units.

Homework
For homework, learners will complete a [Desmos Activity.](#) This assignment has students find missing side lengths in 45°-45°-90° triangles and 30°-60°-90° triangles, by application of special right triangle relationships. The students are instructed to reduce all radicals; a "Great Job" message will appear to students when they respond correctly and a "That is incorrect! Try again!" message will appear if students are not providing the correct side length.

Assessment
The summative assessment for this the Polyhedrons Unit is researching and constructing a polyhedron of choice. Below are the Project resources.
  ● [Polyhedron Project Information](#)
  ● [List of Polyhedrons](#)
  ● [Polyhedron Project Proposal](#)
  ● [Polyhedron Project Rubric](#)

Accommodations
By allowing students to access the [GeoGebra](#) tool, time can be saved for students requiring extended time on activities.

The following accommodations could be provided to meet the needs of all the learners in any classroom:
  ● Instructing this lesson over a three-day period. On the first day, students would learn/review how to simplify radical ratios. On the second day of the lesson, students would explore 45°-45°-90° triangle relationships. On the third day of the lesson, students would explore 30°-60°-90° triangle relationships.
  ● The instructor could provide a list of perfect squares to assist learners in simplifying radicals.
  ● The instructor could decide to keep learners/partnerships at the same pace throughout the lesson, instead of allowing learners/partnerships to progress through the investigations at their own pace.

APPENDIX B: Revised Student Handouts

Name: _____   Date: _____   Bell: _____

Special Right Triangles Investigation (Day 1)

Problem Based off of "Ask, Don't Tell (Part 3): Special Right Triangles" featured in MCTM (June 2015)

Warm Up

---

1. How can you prove $\sqrt{48} = 4\sqrt{3}$?

2. Simplify $\sqrt{63}$.

3. How can you prove $\frac{4\sqrt{3}}{3} = \frac{4}{\sqrt{3}}$?

4. Rationalize $\frac{10}{\sqrt{7}}$.

5. Simplify $3\sqrt{50}$.

Investigation 1: Follow each step, as directed.

---

1. Sketch an **isosceles** right triangle. Label the right angle vertex C and label the other two vertices A and B.
2. Write in the angle measures for all three angles.
3. Pick any integer for the length of the legs. Find the length of the hypotenuse. You can use the Right Triangle Explorer to check your work. Make sure to simplify all radicals.

---

4. Fill in the first row with your triangle's information from #3. Then, find 3 peers and fill in their information.

	Leg	Leg	Hypotenuse
My triangle			

5. What relationships, if any, do you see between the legs and the hypotenuse, in a 45º-45º-90º triangle?

6. Now, sketch another **isosceles** right triangle. Write in the angle measures for all three angles. Label the right angle vertex C and the other two vertices A and B. Pick any integer for the hypotenuse. Solve for the missing leg lengths. You can use the Right Triangle Explorer to check your work. Remember to rationalize and simplify all square roots!
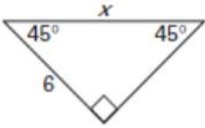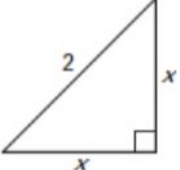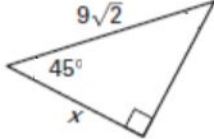
7. Fill in the first row with your triangle's information from above. Then, find 3 peers and fill in their information.

	Hypotenuse	Leg	Leg
My triangle			

8. Using the pattern you see from the chart above, explain how you find the length of the legs of a 45º-45º-90º triangle, if you know the length of the hypotenuse.

45°-45°-90° Triangle Practice

Find the value of x. Write your answer in simplest radical form.

1.	2.	3.

Draw and label a diagram with the given information in each problem, then find the measure of the missing sides. Write answers in reduced radical form.

4. If $\triangle ABC$ is an isosceles right triangle with $m\angle C = 90^o$ and $AB = 12\sqrt{2}$, find AC and BC.

AC=_____ BC=_____

5. If $\triangle ABC$ is an isosceles right triangle with $m\angle C = 90^o$ and $AB = 9$, find AC and BC.

AC=_____ BC=_____

6. If $\triangle ABC$ is an isosceles right triangle with $m\angle C = 90^o$ and $AC = 7\sqrt{2}$, find AB and BC.

AB=_____ BC=_____

7. If $\triangle ABC$ is an isosceles right triangle with $m\angle C = 90^o$ and $BC = 10$, find AB and AC.

AB=_____ AC=_____

8. Find the perimeter of a square whose diagonal length is 15 centimeters.
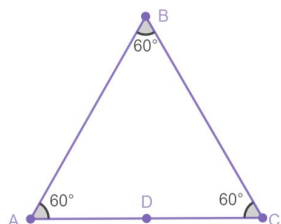
Name: _____   Date: _____   Bell: _____

Special Right Triangles Investigation (Day 2)

Problem Based off of "Ask, Don't Tell (Part 3): Special Right Triangles" featured in MCTM (June 2015)
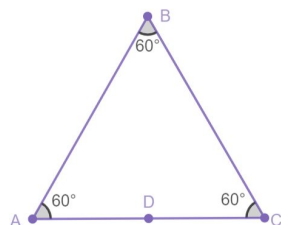
---

1. $\triangle ABC$ is an **equilateral** triangle. Construct a segment from point B to point D, the midpoint of segment AC.

What are the measures of the new angles $\angle DBA$, and $\angle BDA$?

$\angle BAD$	$\angle DBA$	$\angle BDA$
60		

---

2. Segment DB is a perpendicular bisector of $\triangle ABC$. If segment $AB = 2$, find the lengths of DA and DB. Simplify all radicals; it will be easier to see the patterns!

| AB	DA	BD
2		

---

3. Fill in the chart for the missing sides. In $\triangle ABD$, we will call AD the short leg, AB the hypotenuse and DB the long leg. Simplify all radicals. You can use the [Right Triangle Explorer](#) to check your work.
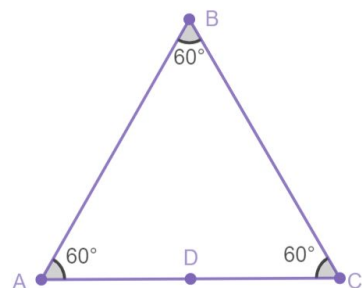
AD	AB	DB
1	2	$\sqrt{3}$
2		
3		
4		
6		
11		
20		

---

4. Describe all the patterns you notice from the table above.

---

5. Complete the chart. Simplify all radicals. You can use the Right Triangle Explorer to check your work.
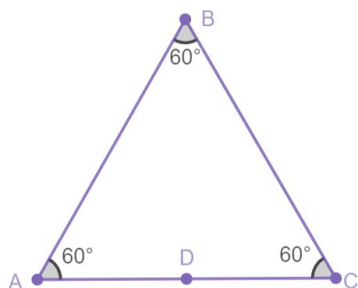
AD	AB	DB
1	2	$\sqrt{3}$
	4	
	10	
3		
6		
		$4\sqrt{3}$
		$7\sqrt{3}$
$2\sqrt{3}$		
		9

6. The short leg of 30º-60º-90º triangle is *n* units. What is the length of the hypotenuse? What is the length of the long leg? How do you know?

Homework:

Complete this Desmos Homework. Below, record any questions you want to discuss with the class tomorrow.