
Mitigating Misconceptions of Preservice Teachers: The Relationship between Area and Perimeter

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Abstract: Preservice teachers often demonstrate misconceptions regarding the non-constant relationship between area and perimeter. The author presents a sequence of hands-on instructional tasks with pentominoes to promote deep conceptual understanding of this relationship.

Keywords: area, pentominoes, perimeter, preservice teachers

1 Introduction

If two shapes have the same perimeter, is their area also the same? This question is included in the third grade mathematics curriculum (see Table 1). I recently posed this question, not to a group of third grade students, but to a class of preservice teachers (PSTs), curious about their understanding regarding the relationship between area and perimeter.

Table 1: Common Core State Standards (NGA/CCSSO, 2010) and the Mathematical Education of Teachers Essential Ideas (CBMS, 2012) related to area and perimeter

CCSS.MATH.CONTENT.3.MD.C.5	Recognize area as an attribute of plane figures and understand concepts of area measurement
CCSS.MATH.CONTENT.3.MD.C.6	Measure areas by counting unit squares
CCSS.MATH.CONTENT.3.MD.C.7.D	Recognize area as additive
CCSS.MATH.CONTENT.3.MD.D.8	Solve real world and mathematical problems involving perimeters of polygons, including exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters
MET II Essential Ideas for K-5 Teachers: Measurement and Data–Illustrative activity 1	Explore the distinction and relationship between perimeter and area, such as by fixing a perimeter and finding the range of areas possible or by fixing an area and finding the range of perimeters possible

Although a few of the twenty-four PSTs were hesitant to answer in the affirmative, a majority were confident in their incorrect conclusion that this relationship is constant. These PSTs will likely teach students about area and perimeter, yet they did not possess a deep understanding of the relationship between these two concepts. I considered how I might best address their misconception

in my geometry content course for elementary teachers and implemented a sequence of instructional tasks that proved successful. Below, I discuss these activities as a means to foster conceptual understanding of area / perimeter relationships.

Based on intuitive reasoning, many students believe that all rectangles of a given area have the same perimeter, or vice versa (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998; Tan Sisman & Aksu, 2016). Misconceptions held by elementary students are sometimes shared by both preservice and in-service teachers, who may themselves struggle with this concept. Ma (1999) found that 90% of the 23 elementary teachers from the U.S. who took part in her study agreed with a student response that the area of a rectangle increases with its perimeter. Livy, Muir, and Maher (2012) extended this work, reporting that 72% of graduating PSTs (n=222) held the same misconception. In this paper, I describe a lesson that helps teachers “robustly understand the mathematical content knowledge for the age groups or grades they may teach,” developing “solid and flexible knowledge of core mathematical concepts and procedures” (AMTE, 2017, pp. 7-8) as they explore various pentomino tasks.

The pentomino activity detailed in this article, which could be used in an elementary classroom, has been adapted for the use of PSTs. These tasks build upon PSTs’ experience exploring area and perimeter of general polygons. The primary objective of this lesson is to provide an opportunity for PSTs to explore the relationship between area and perimeter and to deepen their conceptual understanding of this relationship. To do this, PSTs will construct the various pentomino figures, then calculate both the area and perimeter for each pentomino, as well as for various pentomino combinations.

2 Pentominoes: An Overview

The video game *Tetris* was addictive to many in the 1980s. The designer of *Tetris*, Alexey Pajitnov, was inspired to create the game from his love of a puzzle game known as pentominoes (Brown, 2016), in which wooden shapes needed to be fit together in certain ways. Instead of using pentominoes, consisting of five squares each, Pajitnov decided to use tetraminoes, each composed of four squares, for his game. However, tetraminoes was a mouthful, so the name was shortened to simply Tetris. Thus, a new video game entered the world of entertainment.

Pentominoes and tetraminoes are both subsets of polyominoes. Polyominoes are plane shapes constructed by joining congruent squares together such that each square shares a common side with the adjoining square. Each type of polyomino is given a name based on the number of squares used in its construction. Pentominoes, made popular by Martin Gardner (1957), consist of five squares. Consequently, each pentomino has an area of five square units. Twelve unique pentominoes can be created using these five squares. Although each of these figures has the same area, not every pentomino has the same perimeter. This fundamental property of pentominoes makes this manipulative helpful in teaching about the relationship between area and perimeter.

3 Area and Perimeter Relationships: A Set of Tasks

Hands-on tasks give PSTs an opportunity to explore the relationship between area and perimeter. The activities shared below both deepen PSTs’ understanding of this relationship and model activities that can be used in their own future classrooms. I taught this seventy-five minute lesson to twenty-four PSTs in an undergraduate geometry content course. This entry-level course was designed specifically for PSTs pursuing licensure in elementary education, early childhood education, or special education. The second half of the fifteen-week course, in which this lesson is

situated, is dedicated to geometric measurement concepts, including area and perimeter of polygons. PSTs were all pre-assigned to specific four-member groups. I intentionally assigned at least one PST to each group who has demonstrated conceptual understanding of past topics in geometry and measurement, along with an ability to teach others, having previously found that this distribution helps to keep groups more on task and moving forward with productive group discussion. The other members of the group were randomly assigned.

3.1 Opening the Investigation: Triangles and Rectangles

As PSTs awaited the start of class, I engaged them in an opening activity. Giving each group two pieces of string of equal length, I asked the PSTs to form one string into the shape of a triangle and the second into a rectangle. Each shape should use the entire length of string provided. I then asked the PSTs to consider both the perimeter and the area of each of the two shapes. Is there a relationship between the perimeters and areas of the two shapes created? If so, what is that relationship?

PSTs quickly determined that the perimeters of both shapes must be equal, since the strings are the same length. Most PSTs concluded that the areas of the shapes must also be equal, since they have the same perimeter. A few dissenting voices questioned whether the areas were equal, but none were able to provide reasoning that convinced the majority of their peers.

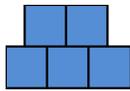
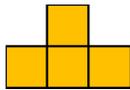
The answer to this question is not trivial, and as the responses of the PSTs indicate, it is not completely intuitive. The PSTs' responses indicate a common misconception, as stated above—that figures having the same perimeter must also have the same area. Instead of providing the correct reply, I asked the PSTs to keep this question in mind as they continued to explore the relationship between area and perimeter in the subsequent tasks.

3.2 Creating and Recording Pentominoes

To begin our exploration of pentominoes, I distributed five square tiles and a piece of grid paper to each PST. I wrote the word "pentomino" on the board, then facilitated a short group discussion of what might be meant by the term. The PSTs were able to make connections both to the number five and to the concept of dominoes. Next, I articulated the rules for creating valid pentominoes. I anticipated that PSTs may misunderstand the rules of creating a valid pentomino and attempt to make pentominoes from the tiles that do not have a fully connecting side, so I displayed a few examples of both valid and invalid pentominoes and had the class as a whole categorize them. These are illustrated in Table 2.

I invited the PSTs to independently explore and investigate the different pentominoes that could be created using five square tiles (see Figure 1). They recorded on grid paper each new pentomino they discovered. Some of the PSTs approached the task strategically, moving certain tiles to create new pentominoes; others approached the task without a set strategy, randomly reconfiguring the tiles to create new pentominoes. As I circulated around the room, I asked questions such as: *How can you arrange the five tiles to make a valid pentomino? If you rotate your figure, will it be the same as another you have already created? If you reflect your figure, will it be the same as another you have already created?* As they continued to work, some of the PSTs recognized that they had inadvertently included the rotation or reflection of a shape as a unique pentomino.

Table 2: *Common invalid pentomino constructions.*

Rules for Creating Valid Pentominoes	Example of Invalid Pentomino
1) the tiles must touch fully on the sides	
2) the tiles may not be connected only at corners or partially on one side	
3) all five tiles must be used	
4) rotated or reflected figures are not unique (i.e., they are congruent)	

As PSTs continued to explore, I asked whether they thought that the number of possible pentomino pieces was finite or infinite. Most thought that there would be a set number of shapes, although there was some disagreement as to what this number might be. I intentionally did not tell the PSTs the total number in advance, in order to provide them with an opportunity for continuing exploration.

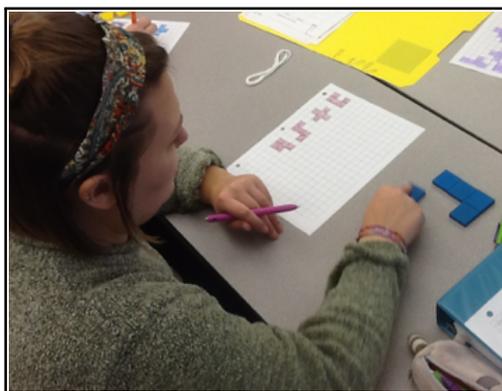


Fig. 1: *PST exploring possible combinations of tiles to create pentominoes.*

Once PSTs had had sufficient time to explore, they were given a few minutes to discuss their results with a partner. Following this, the class took part in a “gallery walk,” at which time everyone left their grids of recorded pentominoes at their tables and slowly walked around the room, briefly examining each of their peers’ work. From this, PSTs often recognized any pentominoes they may have missed. Following the gallery walk, we came to a class consensus that there are exactly 12 possible pentominoes.

As PSTs added any missing pentominoes to their representations, groups discussed how one might easily reference specific shapes. PSTs agreed to use letters of the alphabet to identify various pentominoes. After giving PSTs time to conjecture which letter might best match each shape, I projected a visual of a standard naming scheme, which made use of the letters F, L, I, P, N, T, U, V, W, X, Y, and Z. Figure 2 illustrates one PST’s representations of the twelve pentomino shapes, with letters identifying each.

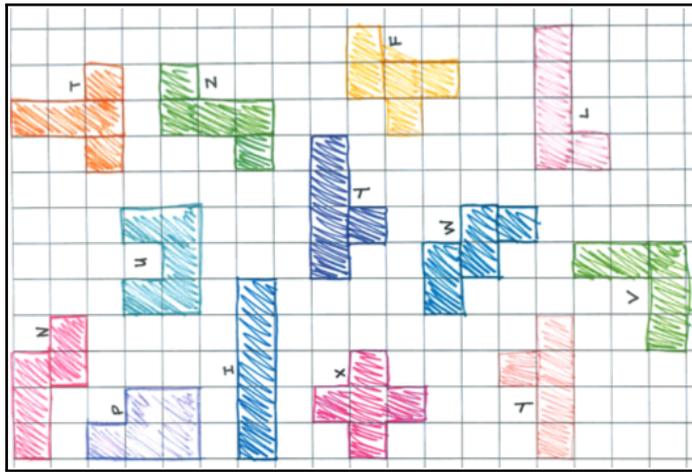


Fig. 2: The twelve pentomino pieces with corresponding letter names.

3.3 The Heart of the Lesson: Area and Perimeter Relationships

To facilitate their continuing exploration, PSTs traded in their square tiles for a set of twelve pentominoes. They were asked to find and record the area and perimeter of each pentomino. The PSTs used “Pentomino Investigation” tables to keep track of their findings. An example table is provided in Figure 3. The area and perimeter of each pentomino can be found either by looking at the two-dimensional drawing of each or by examining the actual manipulative. As Figure 4 suggests, some PSTs touched or drew each side unit as they computed the perimeter, and some marked each square unit as they computed the area.

Name	Sketch	Area	Perimeter	Name	Sketch	Area	Perimeter
T		5 sq. units	12 units	F		5 sq. units	12 units
U		5 sq. units	12 units	P		5 sq. units	10 units
V		5 sq. units	12 units	L		5 sq. units	12 units
W		5 sq. units	12 units	I		5 sq. units	12 units
X		5 sq. units	12 units	Y		5 sq. units	12 units
Z		5 sq. units	12 units	N		5 sq. units	12 units

Fig. 3: Pentomino Investigation table.

Some of the PSTs determined all of the values for area and perimeter without actually computing each individually. Many of these had the misconception that since all pentominoes had the same area, they all also must have the same perimeter, reasoning that area and perimeter must vary simultaneously. One or two of the PSTs initially miscounted the side units or square units, leading to a few incorrect calculations of perimeter or area. After providing sufficient time to complete the table, I asked the PSTs to discuss their findings with their groupmates, focusing particularly on



Fig. 4: PSTs using physical pieces to measure perimeter and area.

whether shapes with the same area always have the same perimeter. Following some discussion, all of the groups came to a consensus that one of the pentominoes (the “P” shape) has a perimeter that is less than that of the other shapes. I then asked the groups to consider why this particular pentomino might have a smaller perimeter than the others, as well as why all of the pentominoes have the same area. Though the class did not yet articulate what these findings meant for their understanding of the relationship between area and perimeter, an element of cognitive dissonance had been introduced for those with original misconceptions.

3.4 More Area and Perimeter Relationships

The final task in this lesson involved combining two (or more) pentominoes to explore the combined area and perimeter, seeking to identify any patterns as well as the greatest and least perimeter possible for a combined figure using two pentominoes. In addition to further solidifying a conceptual understanding of the relationship between area and perimeter, this task also incrementally builds upon initial understanding by illustrating the additive property of area (see Table 1). I used the Puzzling Pentominoes resource from NCTM’s Illuminations (2009) to facilitate this task. PSTs were asked to combine pentominoes (first without using the P shape, then intentionally using the P shape), draw the combined figures on grid paper, and calculate the figures’ area and perimeter. As pentominoes can be combined in a wide variety of ways, the PSTs connected them in many unique constructs as illustrated in Figure 5.

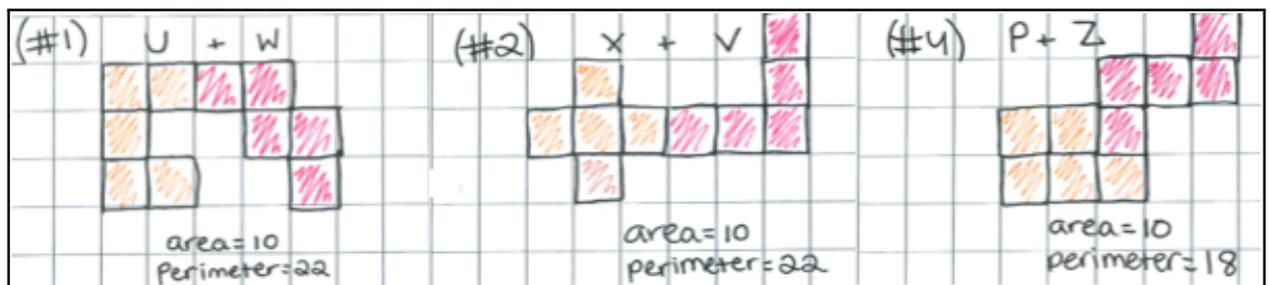


Fig. 5: Various pentomino configurations.

When combining pentominoes, a few PSTs simply added the perimeters of the individual pieces instead of recounting the tile units in the perimeter of the combined figure. Upon comparing results with peers, this mistake was quickly corrected.

After PSTs completed this task, we debriefed as a whole class, and I asked the PSTs to reflect on what they had learned from the day's tasks. The PSTs all indicated that they felt more confident in their understanding of the relationship between area and perimeter. They acknowledged that two figures may have the same area, but not necessarily the same perimeter, and vice versa. Returning to our opening triangle-versus-rectangle activity, PSTs now replied that although the two shapes clearly have the same perimeter, the areas of the two shapes may not be equivalent.

3.5 Extensions

My expectation was that the entire class complete the first half of the Puzzling Pentominoes task, investigating the combined area and perimeter of two adjacent pentominoes. A few groups of PSTs completed this portion of the task before the rest of the class; I asked these PSTs to complete the remaining questions, which asked for generalizations for the area and perimeter of combinations of more than two pentominoes. To facilitate this, PSTs first recorded information about various combinations in a table (see Figure 6), then attempted to generalize a formula that represents the pattern found in areas and perimeters. Although the groups that worked on these questions were able to quickly generalize a formula representing the area of any given combination of pentominoes, they were unable to come to a consensus regarding a general formula for perimeter in the time allotted.

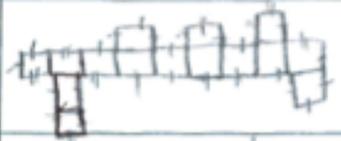
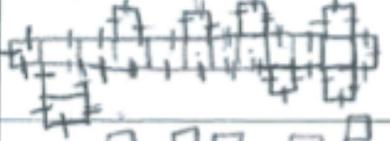
NUMBER OF PENTOMINOES	SKETCH	NAMES OF PENTOMINOES USED	PERIMETER
2		T, U	22
3		T, U, F	32
4		T, U, F, X	42
5		T, U, F, X, L	52

Fig. 6: PST-created table recording perimeters of various combinations of pentominoes.

Other possible extensions include asking PSTs or groups the following questions: *How can you make a rectangle out of different numbers of pentominoes? Can you make a square out of some combination of pentominoes? How can you use all twelve of the pentominoes to create a rectangle with no holes or gaps? What would the total area of this figure have to be? How do you know this?*

Although I implemented this lesson with PSTs, it can be adapted to promote mathematical reasoning for students at every level. Pentominoes develop reasoning skills by providing students with the opportunity to construct arguments, justify decisions made, and critique the reasoning of others. Edwards, Meagher, & Özgün-Koca (2017) developed two pentomino tasks, implemented in a ninth-grade classroom, that focused on developing reasoning. In addition, pentominoes, expanded into the larger family of polyominoes, are a useful tool for introducing elementary number theory to students (Tracy & Eckert, 1990).

4 Conclusion

The relationship between area and perimeter is not fully intuitive; it requires higher-order thinking in order to conceptualize. PSTs may initially possess misconceptions regarding this relationship (Livy, Muir, & Maher, 2012). Inservice teachers may also have similar misconceptions (Ma, 1999); the above tasks may serve as an appropriate professional development activity. The Conference Board of the Mathematical Sciences (2012) insists that “a strong understanding of the mathematics a teacher will teach is necessary for good teaching. Every elementary student deserves a teacher who knows, very well, the mathematics that the student is to learn” (p. 24). Hands-on tasks, such as those described above, can challenge PSTs to develop a conceptual understanding of this relationship, which hopefully they will in turn pass along to their future students. Through the pentominoes tasks, PSTs gained a more robust understanding that even when the area of a figure remains constant, the perimeter may vary. By participating in this activity, PSTs have the opportunity to both deepen their own content knowledge as well as practically gain insights on how they might consider using pentominoes as a learning tool in their own future classrooms.

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