The Joy of Following Students Down Unexpected Paths

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Abstract: *As our students think and reason mathematically, sometimes we need to follow their paths, even if they are different than those we have anticipated they will take. Occasionally these unexpected paths can lead to surprising connections. In this article, the author describes new paths her class took to generalize the sum of the interior angles of convex polygons.*
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1 Introduction

Creating a classroom in which students are actively engaged in mathematical thinking, rather than passively accepting what an instructor shares with them, can be a challenge. For instance, when instructors are successful at fostering a setting in which students have freedom to explore mathematics, they may take a direction that was not anticipated by the instructor. This can be a joyful time for both the students and instructor, alike. This article highlights what happened when a class of preservice teachers was allowed to follow their own paths as they explored the sum of the interior angles of convex polygons. While this article discusses university students engaged in mathematics, this type of engagement can, and should, be encouraged among students at all levels.

1.1 The Classroom

The class consisted of 16 preservice middle school teachers in their second year. In mathematics content courses, I challenge my students to delve more deeply into mathematical concepts than they have previously. In addition to focusing on mathematics content, I structure my courses to further develop candidates' reasoning habits. Specifically I "resist the urge to tell students how to solve a problem," and I expect my students to "communicate their reasoning to their classmates" (NCTM, 2009, p. 11).

Typically, classes begin with an activity. Teacher candidates are encouraged to focus on engaging in the mathematical practices described by Ohio's Mathematics Standards. This is done both to familiarize the students with the practices and to hone their mathematical thinking skills, as the practices are relevant to all mathematics students. That is, I encourage teacher candidates to:

- Make sense of problems and persevere in solving them;
- Reason abstractly and quantitatively;
- Construct viable arguments and critique the reasoning of others;

- Model with mathematics;
- Use appropriate tools strategically;
- Attend to precision;
- Look for and make use of structure;
- Look for and express regularity in repeated reasoning (Ohio Department of Education, 2010).

When the students arrive at what they believe is the end of a problem, I remind them to think like a mathematician and look for other solutions (Polya, 2004).

1.2 The Activity

In a previous class meeting, we had discussed the sum of the interior angles of a triangle and established that this sum is always $180°$. The goal for the current class was to establish that for any convex polygon, the sum of the interior angles is given by $S = 180°(n-2)$, where n is the number of sides of the polygon. Most textbooks state this as a theorem, with some providing an accompanying figure of a "triangulated" polygon—one with all possible diagonals drawn from one vertex (see Figure 1). My goal for the students was to have them explore different convex polygons, notice patterns, make and discuss conjectures, and justify their conclusions. They were not provided with the angle sum theorem or illustrations from school mathematics texts.

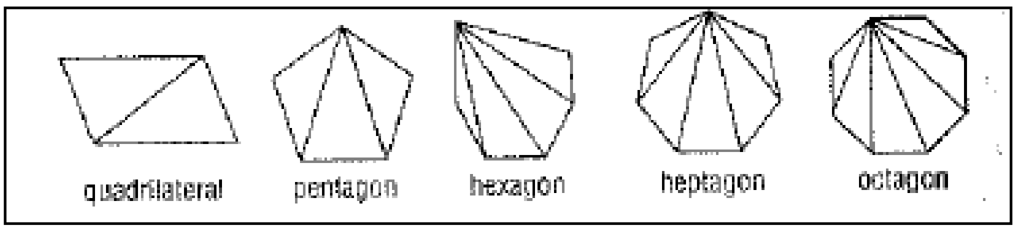

Fig. 1: *Image from textbook Geometry (Boyd, et al., 2004, p. 404).*

Students brought either a laptop or tablet and used the dynamic geometry software, GeoGebra, to aid in their investigations. We began by exploring quadrilaterals as a group. After all students had used GeoGebra to sketch a few different convex quadrilaterals and drawn a diagonal to divide each quadrilateral into two triangles (thus discovering that the sum of the interior angles of a quadrilateral is $360°$), they explored convex polygons with more than four sides in small groups.

After some time had passed, students reconvened in a large group to share what they had found. I anticipated that each group would triangulate their polygons (as textbooks had in Figure 1) and generalize the patterns they saw. Instead, teacher candidates found different ways to divide the polygons, which led to numerous other conjectures and patterns.

2 Student Findings

The first group shared that they had taken each polygon and divided it into one triangle and one other polygon. They explained that by doing this, they were able to see that polygons with n sides are composed of a triangle and a polygon with $(n-1)$ sides (See Figure 2).

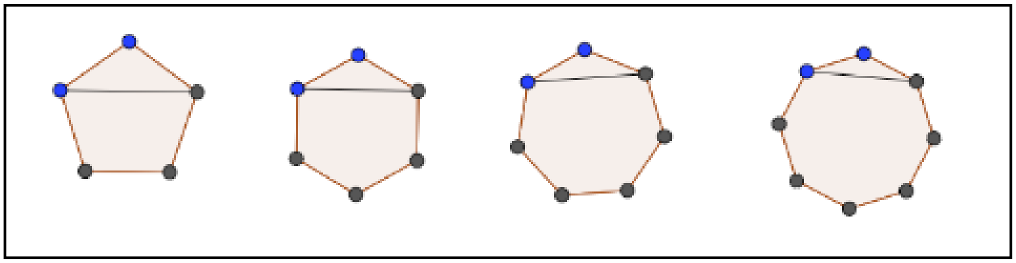

Fig. 2: *Polygons with n sides decomposed into a triangle and a polygon with $n-1$ sides.*

Another group quickly took the floor and shared how they had divided their shapes. Using regular polygons, they explained that they had drawn a diagonal that "cut the shape kind of in half." That is, if the regular polygon had an even number of sides, they used a line of symmetry as a diagonal, and if the polygon had an odd number of sides, the diagonal was "almost" a line of symmetry (See Figure 3).

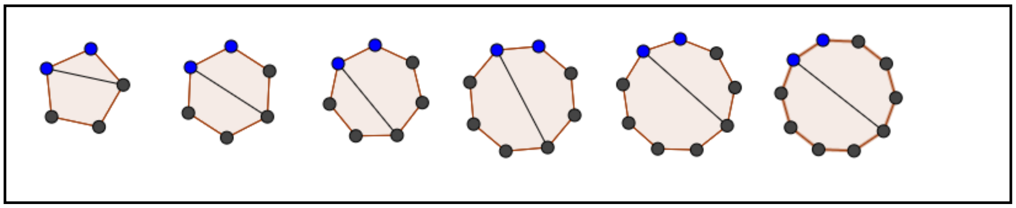

Fig. 3: *Polygons cut "kind of in half."*

Other groups of teacher candidates had divided the polygons in a similar manner (sometimes a combination of the two methods), but no group triangulated polygons in the manner suggested by school texts.

3 Following the Students

Rather than directing the students to triangulate each polygon, we further explored, as a class, the two approaches offered by the two groups of candidates.

3.1 Analysis of "Triangle Plus Polygon" Figure

First, I asked teacher candidates to report patterns they noticed in Figure 2 (i.e., the "triangle plus polygon" sketch). They began by creating a chart to summarize the information they found in their GeoGebra sketches. As Table 1 suggests, candidates worked recursively to fill in their table—that is, they used angle sums from completed rows to deduce sums in subsequent rows.

Table 1: *Student Chart Summarizing Polygon + Triangle.*

Number of Sides (n)	Shapes	Sum of Interior Angles
5	Quadrilateral + Triangle ($360° + 180°$)	$540°$
6	Pentagon + Triangle ($540° + 180°$)	$720°$
7	Hexagon + Triangle ($720° + 180°$)	$900°$
8	Heptagon + Triangle ($900° + 180°$)	$1080°$

Candidates noticed that the sum increased by $180°$ for each additional side of the polygon. This recognition led naturally to a review of arithmetic sequences. Students discussed that the sum of the interior angles for a convex polygon with n sides could be found using the recursive equation $a_n = a_{n-1} + 180°$; $a_3 = 180°$ or the explicit equation $a_n = (n-2) \cdot 180°$; $n \geq 3$. After this, the class seemed satisfied with the analysis of the chart, so we moved onto the next figure.

3.2 Analysis of "Kind of in Half" Figure

Again, candidates began by making a chart while discussing different patterns they had found (See Table 2). Based on the new chart, one group created two cases: (1) if the polygon had an odd number of sides, it could be divided into two polygons, one with an even number of sides and one with an odd number of sides; and (2) if the polygon had an even number of sides, it could be divided into two polygons, each with the same number of sides.

Table 2: *Table Summarizing Shape Cut "Kind of in Half."*

Number of Sides (n)	Shapes	Sum of Interior Angles
5	Quad + Triangle $(360° + 180°)$	$540°$
6	Two Quads $(2 \cdot 360°)$	$720°$
7	Quad + Pentagon $(360° + 540°)$	$900°$
8	Two Pentagons $(2 \cdot 540°)$	$1080°$
9	Pentagon + Hexagon $(540° + 720°)$	$1260°$
10	Two Hexagons $(2 \cdot 720°)$	$1440°$
11	Hexagon + Heptagon $(720° + 900°)$	$1620°$

The rest of the class picked up this line of reasoning—with one student suggesting the construction of a general equation for the sum of the interior angles from the new pattern—similar to what was done with the first chart. This proved to be a difficult task. Teacher candidates generated a number of conjectures—some true, others not. For example, one group initially thought that if n was an even number, the polygon was a combination of two polygons with $n-2$ sides each. After further study, candidates recognized that this conjecture was only true for their data when $n = 6$. After more struggles, one group shared their approach to the problem. They created a third chart similar to the two previous ones, but added a new column counting the number of vertices of the polygons as they related to the shapes that formed the polygon (See Table 3).

Table 3: *Table with Number of Vertices Information.*

Number of Sides (n)	Shapes	# of Vertices	Sum of Interior Angles
5	Quad + Triangle $(360° + 180°)$	4+3	$540°$
6	Two Quads $(2 \cdot 360°)$	4+4	$720°$
7	Quad + Pentagon $(360° + 540°)$	4+5	$900°$
8	Two Pentagons $(2 \cdot 540°)$	5+5	$1080°$
9	Pentagon + Hexagon $(540° + 720°)$	5+6	$1260°$
10	Two Hexagons $(2 \cdot 720°)$	6+6	$1440°$

The additional column, # of Vertices, held the key the students needed to generalize the pattern they observed. Candidates noticed that the sum of the vertices of the composing polygons was always two more than the number of sides (or vertices) of the main polygon. The generalized rule they created had two parts and used the greatest integer function.

- If n were even, then the polygon could be created using two polygon with $\frac{n+2}{2}$ sides each.
- If n were odd, then the polygon could be created using two polygons with $\lfloor\frac{n+2}{2}\rfloor$ and $\lfloor\frac{n+2}{2}\rfloor+1$ sides, respectively.

Students acknowledged that this rule did not tell them the sum of the interior angles of the polygon; rather it provided them with a way to decompose the polygons into ones with fewer sides.

4 Where are all the Triangles?

At this point, I sensed the class was content with their work and appeared ready to move on to the next topic. I was pleased with the connections the class had made to sequences and the greatest integer function, but we had not yet broken the polygons into triangles, as they would see in traditional textbooks. I still did not want to explicitly instruct them to use triangles, looking instead for a way to build on the ideas they had already explored. I asked the class the following question: *If I have a polygon with 12 sides, what is the sum of the interior angles?* The class used the information from the first chart and the explicit equation and decided upon the answer of $a_{12} = (12-2)\cdot 180° = 10\cdot 180° = 1800°$. I asked them how they would use the second chart to arrive at a solution. Using this method, they said that a polygon of 12 sides (a dodecagon) would split into two polygons, each with seven sides, so according to their chart, that would be $S = 900° + 900° = 1800°$. I pressed them further by asking what they would do if the second chart were not "expanded." For instance, without the additional column, could they use the table to determine the sum of the angles in a dodecagon? Tackling my question, they went to work and created a chart that showed the breakdown of the dodecagon into polygons with fewer and fewer sides (See Figure 4).

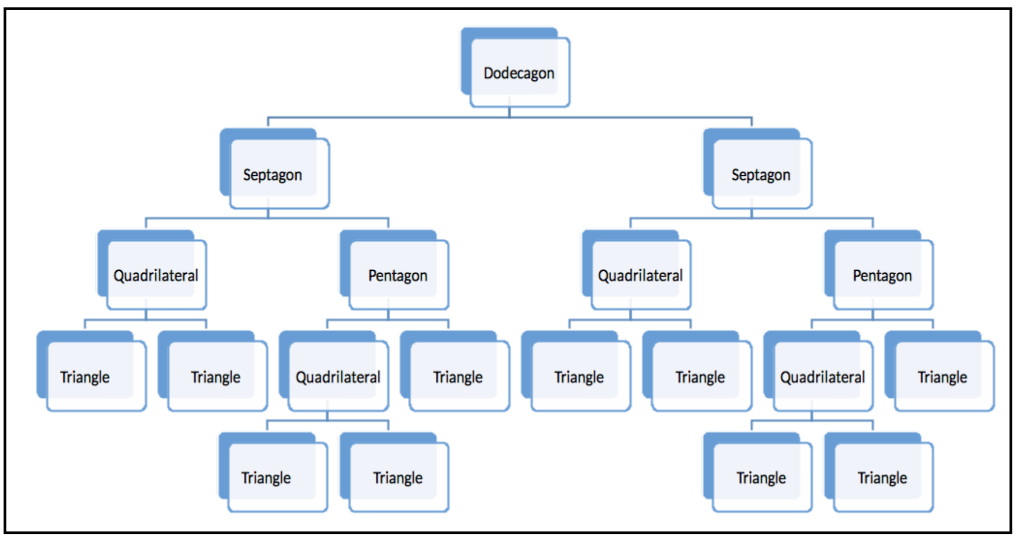

Fig. 4: *Chart for dodecagon.*

When I asked the class if they saw any connection between their chart and calculation, $a_{12} = 10\cdot 180° = 1800°$, they were quick to see the ten triangles that composed the 12-gon. Further discussion led to the conclusion that any convex polygon could be broken into $n-2$ triangles, which offered another justification for the textbook formula for the sum of a polygon's interior angles, $S = 180°(n-2)$. I challenged the class to use GeoGebra to demonstrate the 12-gon divided into 10 triangles and, as before, a variety of divisions was created (See Figure 5). Class ended with an examination of triangulation pictures from a few textbooks and a wrap-up of the mathematics that had emerged during our class meeting.

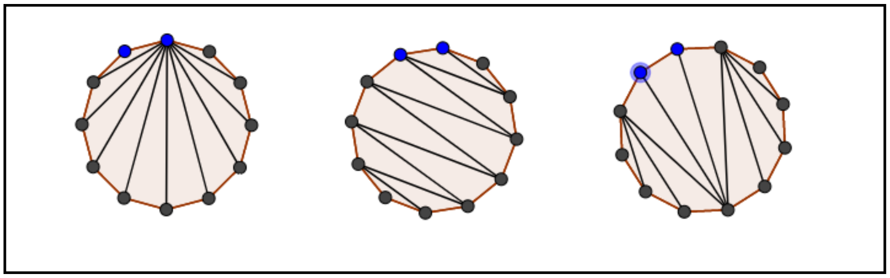

Fig. 5: *12-gon divided into 10 triangles.*

5 Conclusion

The class did arrive at the anticipated conclusion that the sum of interior angles of a convex polygon with n sides is given by $S = 180°(n-2)$ and that this can be modeled by choosing a single vertex and drawing all diagonals from that point. However, we took what I call the 'scenic' route, and found more than one way to convince ourselves of this fact. The class engaged in many of the mathematical practices (construct viable arguments, critique the reasoning of others, look for and make use of structure) suggested by NCTM and ODE. By allowing the students to take the lead, they engaged in mathematical reasoning (exploring, conjecturing, justifying) and had the added bonus of traveling down a path other than the one mapped out for them by the textbook. I do not expect the next set of students to necessarily travel this same path, but I do expect to experience once again the joy of mathematical discovery as they follow their own road.

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