Pixels: Creating Lessons with Historical Connections Between Perimeter and Circumference

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Abstract: *When teachers play the believing game (Elbow, 1986) while in-the-moment of teaching, they attempt to tease out the merit in students' answers or comments that might, at first, seem incorrect. Retrospective analysis of video of classroom episodes can allow mathematics teachers to play the believing game after-the-moment of teaching. Within this narrative we share how playing the believing game while analyzing video led us to explore the merit of one student's comment and the historical connections between circumference and perimeter. Additionally, we describe an activity that Beth created for students in a subsequent mathematics course in order to help preservice teachers see those historical connections.*

Keywords: *Discourse, video analysis, math history, geometric measurement*

1 Introduction

We begin with a vignette from a course for future elementary and middle grades mathematics teachers.

TEXTBOOK EXERCISE : A regular 20-gon and a regular 40-gon are inscribed in a circle with radius of 15 units. Which of the perimeters is closest to the circumference of the circle? Why do you think that is? (Retrieved from: `www.ck12.org/book/CK-12-Geometry-Second-Edition/section/10.6/`)

BEN : Could you technically do $2\pi r$ for like any polygon? Like the way I'm thinking of it is sort of like a TV, like the triangles would be like the most basic pixel, you know what I mean? As you continue on the spectrum [from an equilateral triangle to square to any regular n-gon], you know, the circle becomes like a super high resolution, super small pixel. So could you like essentially like, you know, treat these polygons as just I guess as low resolution circles. (Classroom Conversation, April 17, 2014)

After watching video of Beth's mathematics classroom and subsequently replaying the video several times, we became curious about Ben's comment. His comment led us to engage with the mathematics that he suggested and explore the relationship between the circumference of a circle and perimeters of different regular polygons with altered "diameters" (Harkness & Noblitt, in press). After this exploration we investigated research about historical connections to geometric concepts of perimeter and circumference, which is the focus of this article. Using narrative, we "story" (Clandinin & Connelly, 1990) the teaching episode, our own research into the history of a mathematical topic, and how our curiosity influenced Beth's subsequent teaching.

2 Setting

2.1 Course and Students

The course in which Ben's comment (see above) occurred was *Mathematics for Elementary and Middle Grades Teachers* at a Midwestern university; Beth, was the instructor. The overall goal of the course was for students to develop deep understanding of mathematical concepts important to teaching elementary and middle grades mathematics. About one-fourth of the 28 students were planning to become middle grades teachers and most all students were sophomores and juniors. The content of the course was geometry; topics included two- and three-dimensional figures, measurement, congruence, and transformations.

2.2 Believing and Doubting Research

Shelly was an observer of the class meetings, looking for specific moments in class when Beth was playing the believing game with her students. With Shelly's research background in the believing game in the mathematics classroom, she was able to identify moments of believing and doubting (Elbow, 1986; 2006) while taking field notes. We recorded class sessions in order to have a record of these moments to refer back to during our data analysis. Very briefly, when a mathematics teacher hears students' answers or conjectures she deems as incorrect, in order to believe, she must suspend her own logic and assumptions and attempt to tease out what might be correct in students' answers or conjectures. It is paradoxical because she must doubt her own mathematical understanding as the only understanding and believe there is merit in students' thinking or answers and that she can learn mathematics from her students. When teachers doubt, they typically try to find the flaws or errors in students' answers or conjectures and are less open to learn from their students. Elbow contended that teachers should balance their practices of believing and doubting.

Using students' conjectures to promote historical connections can be challenging for teachers who have little or no background in mathematics history to draw upon; however, as teachers it is never too late to learn. Playing the believing game (Elbow, 1986; 2005; Harkness, 2009; Harkness & Noblitt, 2017; Noblitt & Harkness, 2017), or attempting to believe rather than doubt students' comments or answers that at first seem wrong or incorrect or illogical, can occur in the moment of teaching or also in hindsight, while watching video of recorded teaching episodes. Doing so retrospectively led us to explore the historical connections inherent in Ben's comment.

2.3 The Lesson

The topic for this lesson was the development of the formula for the area of a circle. First, Beth drew a series of regular n-gons on the board with increasing n values. She referenced those drawings and made the following comments:

> We've talked about n-gons, where n is the number of sides in the polygon . . . If I kept increasing n, as n gets bigger, and by bigger, I mean really big, like a 100-gon; a 1,000-gon; a 1,000,000-gon; a 1,000,000,000-gon . . . what is happening with these n-gons? What are they going to look like? [A student replied, "Circle."] Don't you think so? I mean think about what a 1,000,000-gon is going to look like. Are you going to be able to see

> those segments? Probably not. It's going to basically *look like* a circle. So as n gets bigger, I'm going to say that [the n-gon] "becomes" a circle ... Does that n-gon really become a circle? No. It doesn't really become a circle ... essentially this figure ends up *looking like* [a circle] (Classroom Conversation, April 17, 2013).

After a brief discussion about the ratio relationship that yields pi, Beth drew a circle with an inscribed hexagon on the whiteboard, labelled the side lengths of the hexagon as s, and discussed how to find the area of the hexagon by adding the areas of the triangles together. She then broke the hexagon up into six equilateral triangles and, after a brief discussion of the figure, she reminded the students that the hexagon could have been any regular polygon. She said, "so that perimeter ... it's the perimeter of a regular n-gon. I put n there because I want you to remember that I chose six but it could have been any n.

At this point, Beth felt it necessary to remind students where they were headed with this — trying to derive the formula for the area of a circle. She said,

> Remember $\cdots$ I'm doing this because I want to find the area of a circle. What do you think I want to say about that n in order to get me to a circle? I want n to get bigger and bigger and bigger. So Im going to say as n gets bigger, the ... n-gon "becomes" a circle. "Becomes" is in quotes, because a hexagon doesn't become a circle. So as n gets bigger and bigger, this is not $6s$ anymore ... I would no longer say this is the perimeter ... but the what? The circumference. So I don't write the perimeter of a circle as the number of sides of a circle times what the side length is because there are no sides to a circle [1] ... But we know this circumference divided by diameter is pi, right? So what do we know the circumference is equal to? Pi times the diameter. And many of you have probably seen that written as ... $2\pi r$ (Classroom Conversation, April 17, 2013).

It was then that Ben made the comment in the introduction above (please reread it now). Beth replied:

> Not really. [Please note: *Beth was doubting here. She believed her own mathematical understanding of the situation, which did not include thinking of the n-gons as pixels or using $2\pi r$ to find the perimeter of an n-gon.*] It doesn't work perfectly because ... here are two reasons that I can think of. One, I don't know what the radius is [of an n-gon] and that's what r is. And two, this relationship holds for circles but what does that mean for a polygon? ... Circumference I guess you could say the measurement around. Okay. Fine. But the diameter [of a polygon] ... Like, what does that even mean? [Please note: Beth was discussing her own understandings and assumptions. However, she was asking Ben somewhat rhetorical questions in a manner to convince him of the flaws in his argument] (Classroom Conversation, April 17, 2013).

Unfortunately, the real life constraints of teaching a sixteen-week class to a group of 28 students prohibited Beth from having the freedom and forethought to following up with Ben's comments beyond the brief exchange above. The good news is, as teachers, we can continue to learn from each interaction we have with students and hope that we improve our practice year after year, class after class, student after student. Video can be very beneficial in this respect. After watching the class video, we determined that Beth seemed to predominantly doubt Ben's connection between perimeter and circumference but modestly pursued his notion of using $C = 2\pi r$ to find the perimeter of any n-gon. Was Ben, in fact, describing his own "sticking point" (Ernest, 1998, p. 26) related to the connection between perimeter and circumference? This sticking point had definite historical connections but we needed to do some research.

[1] Editor's note: The number of sides of a circle is a topic of great interest and considerable debate that has more than one correct interpretation. See `https://en.wikipedia.org/wiki/Monogon` and `http://mathworld.wolfram.com/Circle.html` for more details

3 Mathematics and Historical Connections Literature

When teachers use historical connections, the "myth of mathematics as a perfectly finished body of knowledge is challenged" (Ernest, 1998, p. 25). Historical connections can underscore multidisciplinary and interdisciplinary connections. Additionally, relating mathematics to its historical progression can emphasize the natural order of development of the discipline and reveal the "sticking points" (Ernest) inherent in its inventions and conventions. For example, the concept of zero "took a long time to develop and this is still the source of many learners' problems" (Ernest, p. 26) and invention of negative numbers was contentious and accepted, at first, with reservations.

For students who have typically avoided mathematics, Buerk (1985) developed strategies to enhance the learning of the subject. One such strategy is to include the historical perspectives so that students see mathematics as a human endeavor (Bidwell, 1993; Borasi, 1986; Gulikers & Blom, 2001). Students can also see that:

> The development of the mathematical understanding of an individual follows the historical developments of mathematical ideas. The task of education is to make the mind of the pupil go through what his [sic] earlier generations have experienced, to pass rapidly to certain stages, but not to omit any (Gulikers & Blom, 2001, p. 225).

Gulikers and Blom noted that historical connections can help students recognize: (a) how mathematical concepts have developed over time; (b) the non-linear progressions of mathematics development; and, (c) the paradox between the conventions and creativity inherent in mathematics. However, teaching historical topics in mathematics classrooms is difficult because it forces mathematics educators to either "trivialize" the mathematics or to "distort" the history (Fried, 2001, p. 391). Teaching is always a balance, and we contend that addressing the history of mathematical topics despite the possibility of trivializing or distorting that history might have some merits if students understand both the mathematics and mathematics as a human endeavor that is constantly being reinvented.

4 Perimeter and Circumference

More than 4000 years ago Egyptians were exploring connections between perimeter and circumference (Burton, 1995):

> The mathematical papyri that have come down to us contain numerous concrete examples, without any theoretical motivation or prescription-like rules, for determining areas and volumes of the most familiar plane and solid figures. Such rules of calculation must be recognized as strictly empirical results, the accretion of ages of trial-and-error experiences and observations (p. 51).

We used trial-and-error to explore Ben's conjecture (Harkness & Nobitt, in press) and then conducted research about historical connections. According to the website, *Better Explained* (`http://betterexplained.com/articles/prehistoric-calculus-discovering-pi/`), "Our equations don't need to be razor-sharp if the universe and our instruments are fuzzy . . . Whether making estimates or writing software, perhaps you can start with a rough version and improve it over time, without fretting about the perfect model (it worked for Archimedes)" (p. 9). Archimedes used a system of finding the perimeters of inscribed and circumscribed regular polygons to explore the connections between perimeter and circumference (see Figure 1 and Appendix).

He began with a circle with diameter of one unit and constructed inscribed and circumscribed hexagons. Archimedes juxtaposed into Ben's modern-day world, might have, in fact, thought of polygons as "low resolution circles" and asked, "... Could you essentially like treat these polygons as low resolution circles?" Archimedes used the perimeters of these inscribed and circumscribed hexagons to estimate the circumference of the circle with diameter of one, or in this case pi, to be between 3 and ≈3.36 units. Applying an iterative process and increasing the number of polygonal sides from six to 96, Archimedes made improved estimates, eventually calculating the ratio of the diameter of the circle to its circumference, to be between ≈3.143 and ≈3.142 units. Over time, others have attempted to compute pi with astonishing results. In November, 2016, Peter Trueb used y-cruncher$^{\text{TM}}$, a computer program, to find the first 22.4 trillion digits of pi (`www.numberworld.org/y-cruncher/`) which took 105 days.

5 A New Lesson

After doing this historical research, we wondered how we might apply what we had learned to the classroom. Even though the class in which Ben was a student had ended, Beth taught a different course, *Introduction to Higher Mathematics*, the content of which focused on proof techniques. In order to prepare her students for the proof that is irrational, she asked, "What is an irrational number?" During the subsequent discussion, the irrational number pi came up. When asked where pi came from, the students replied that it had something to do with circumference and diameter, but they did not know how the approximation of pi was determined. This reminded Beth of her earlier conversation with Ben and our subsequent research about Archimedes and his approximation of pi. In order to facilitate the students' exploration of Archimedes' work, Beth created an activity [see Appendix] for them to complete, with the goal of gaining an understanding of Archimedes' approximation of pi. The lesson asked students to inscribe and circumscribe some regular polygons. Sketches of inscribed and circumscribed polygons from a sample student are provided in Figure 1.

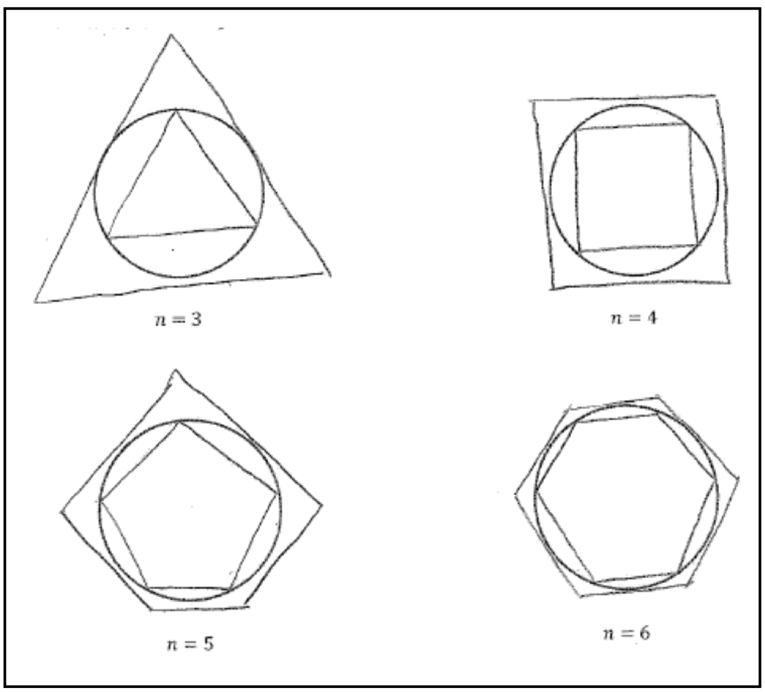

Fig. 1: *Example of student work (see Appendix).*

The activity also asked students to explain what happened to the polygons as the number of sides increased. Finally, the last question probed: "Archimedes used the ideas in #1 and #2 as part of his technique for approximating the value of pi. What do you think he might have done to actually find his approximation of pi?" Several students explained, in writing, how Archimedes might have used inscribed and circumscribed polygons to approximate pi. One student answered,

> As the number of sides increases, the perimeters of the circumscribed and inside the circle will approach the actual circumference of the circle. He could've taken shapes with higher and higher number of sides around a circle with a known diameter. He then found an overestimate and underestimate for pi by finding the ratio of the perimeter of the polygons to the diameter of the circle. The estimates would get closer and closer to pi.

Another student noted,

1. Continue to increase n until edges are unidentifiable;
2. Measure around the shapes (edge $\times n$) and then divide by the diameter;
3. Average overestimate and underestimate. Get closer and closer to pi as n increases.

And, a third student wrote,

> Measures perimeter of inscribed and circumscribed polygons, sees how that measurement relates to diameter of circle, figures how [sic] approximation of pi.

After giving students time to think about the activity and write their responses, Beth engaged the class in a discussion about their answers, having students read what they had written for what they thought Archimedes had done to approximate pi. These answers stood out among the rest as answers that showed real insight into Archimedes' process. After the class discussion, Beth believed that the activity had prompted good discussion and had given students some historical perspective on the approximation of pi.

6 Conclusion

As teachers, we can create lessons which highlight the history of mathematics from typical textbook exercises such as the one at the beginning of this article (above Ben's comment). However, as revealed in this narrative, playing the believing game (Elbow, 1986) while retrospectively watching video of a classroom episode in Beth's classroom led us to first explore the mathematics inherent in Ben's comment (Harkness & Noblitt, in press). As a result, we next investigated research about the historical connections between perimeter and circumference. Subsequently, Beth created the activity, "Archimedes: Approximating π," [see Appendix], to provide students in another mathematics course she was teaching some historical perspective and to highlight the connection that Ben was trying to make between perimeter and circumference.

According to Fried (2001), having students complete such an activity might trivialize the mathematics of Archimedes or distort the history of the approximation of pi, but we contend that it could be a starting point for deeper understanding of the mathematics and of mathematics as a human endeavor. After all, as Guilkers and Blom (2001) wrote, "The task of education is to make the mind of the pupil go through what his [sic] earlier generations have experienced, to pass rapidly to certain stages, but not to omit any" (p. 225). This activity was a modest attempt at providing students with the opportunity to experience the mathematics of Archimedes' approximation of pi. We do not claim that the students actually experienced the same grappling that Archimedes likely experienced. However, it is our hope that they might have gained a glimpse of the approach he used. It is important that students see mathematics as created and consistently recreated by humans throughout the ages. Many thanks to Ben for his insightful comment which allowed us to explore the connections between pixels and circumference and perimeter.

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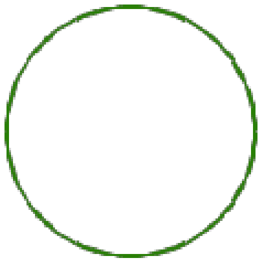
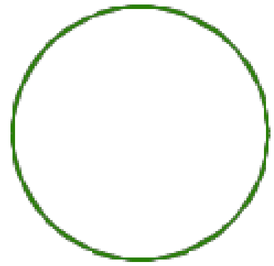
7 Appendix - Student Worksheet

Archimedes: Approximating π

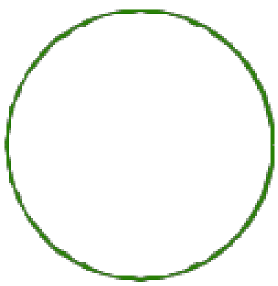
1. For each of the given circles and the given n sketch an equilateral n-gon inscribed in the circle and an equilateral n-gon circumscribed around the circle.



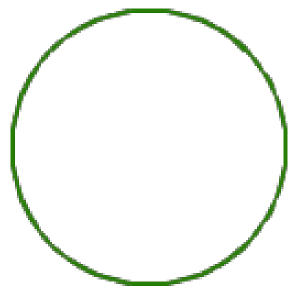
$n=3$



$n=4$



$n=5$



$n=6$

2. What will happen to the inscribed and the circumscribed n-gons as n gets bigger and bigger?

3. Archimedes used the ideas in #1 and #2 as part of his technique for approximating the value of π. What do you think he might have done to actually find his approximation of π? (He did not use Calculus.)