# Stocking Fish: A Recursive Problem

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***Abstract:*** *In the following paper, the authors share the* Trout Pond Exploration*, an activity designed to engage students in sequential reasoning. The authors revise the exploration to include student access to advanced digital technologies, namely spreadsheets. Using these tools, students follow their natural inclination to solve the task using recursive methods.*

***Keywords:*** *Technology, discourse, problem solving*

## 1 Introduction

Flexible reasoning about number sequences — both recursively and explicitly — is an important part of the reform mathematics curricula (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; National Council of Teachers of Mathematics (NCTM), 2000). The *Common Core State Standards* suggests that students "build a function that models a relationship between two quantities [and] determine an explicit expression, a recursive process, or steps for calculation from a context" (CCSS, 2010, p. 70). NCTM (2000) echoes the need for students to represent relationships with iterative and recursive expressions. However, the majority of algebra curricula focus explicitly on defined expressions and functions (Lannin 2004). For the most part, traditional algebra tasks ask students to generalize patterns with an explicit expression. Even though students are encouraged to pay attention to the relationship between two consecutive terms items in a pattern, the focus is always on defining the relationship explicitly. Lannin (2004) provides a rationale for this avoidance in the algebra curriculum, namely that calculations, (especially by hand) are cumbersome, but are necessary, when working recursively.

Lannin (2004), Lannin, Barker, and Townsend (2006), and Rubenstein (2002) agree that examining a pattern or relationship recursively is generally the first inclination for many students. Lanin (2004) states that "students naturally reason recursively when they begin to examine patterns. Recursive reasoning uses an established mathematical relationship between a previous term or terms in a sequence" (p. 217). In this paper, we share an activity – *The Trout Pond Exploration* – that capitalizes on students' first inclination to reason recursively, deploying advanced digital technology (in the form of a spreadsheet).

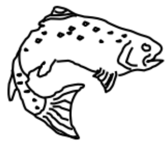
## 2 The Activity

We selected the *Trout Pond Exploration* from the NCTM *Illuminations* resource. The exploration provides an ideal problem scenario for nurturing students' recursive reasoning (see Figure 1). The first question of the task asks students to make a prediction about how the population will grow:

## Trout Pond Exploration

NAME ______________________

Each spring, a trout pond is restocked with fish. That is, the population decreases each year due to natural causes, but at the end of each year, more fish are added. Here's what you need to know.



- There are currently 3000 trout in the pond.
- Due to fishing, natural death, and other causes, the population decreases by 20% each year, regardless of restocking.
- At the end of each year, 1000 trout are added to the pond.

**Fig. 1:** *Trout pond problem (available at* `http://bit.ly/troutpond`*).*

"Do you think the population will grow without bound, level off, oscillate, or die out? Explain why you think your conjecture is reasonable." It is important to give students the opportunity to think carefully at this point and to make a genuine prediction with justification without excessive calculation. There are plausible conjectures for all possibilities at this stage. Some students will argue that the decay 600 (20% of 3000) is smaller than 1000 (the replenishment rate) and so the population will grow without bound. Others will argue that, with the population growing, the 20% will become larger than 1000 and, therefore the population will die off. Others will argue that the decay rate will become larger than the replenishment amount but that, as the population decreases the replenishment amount will again become larger than the decay rate and so the population will oscillate, possibly, honing in one value. Interestingly, in this initial phase, it is rare for students to argue that the population will increase continuously but with an (asymptotic) bound.

Then students are asked to fill out a table (see Figure 2). While we appreciate that filling this table definitely helps students' recursive thinking, we also think that it is very cumbersome and, as will be discussed below, the time spent filling out this table could be spent differently and more efficiently.

**2.** Use the table below (or some other method) to test your conjecture.

| YEAR | NUMBER OF TROUT IN POND | YEAR | NUMBER OF TROUT IN POND |
|---|---|---|---|
| 0 | 3000 | 13 | |
| 1 | | 14 | |
| 2 | | 15 | |
| 3 | | 16 | |
| 4 | | 17 | |
| 5 | | 18 | |
| 6 | | 19 | |
| 7 | | 20 | |
| 8 | | 21 | |
| 9 | | 22 | |
| 10 | | 23 | |
| 11 | | 24 | |
| 12 | | 25 | |

**Fig. 2:** *Trout pond table.*

The last two questions of the task ask students:

- Is it possible to predict the population of the pond after a given number of years? How might you make such a prediction?
- Let the word NEXT represent the population next year, and NOW represent the population this year. Write an equation using NEXT and NOW that represents the assumptions given above.

In the next section, we will share how we adapted this task with the use of a spreadsheet and how this revision made asking higher-order thinking questions possible.

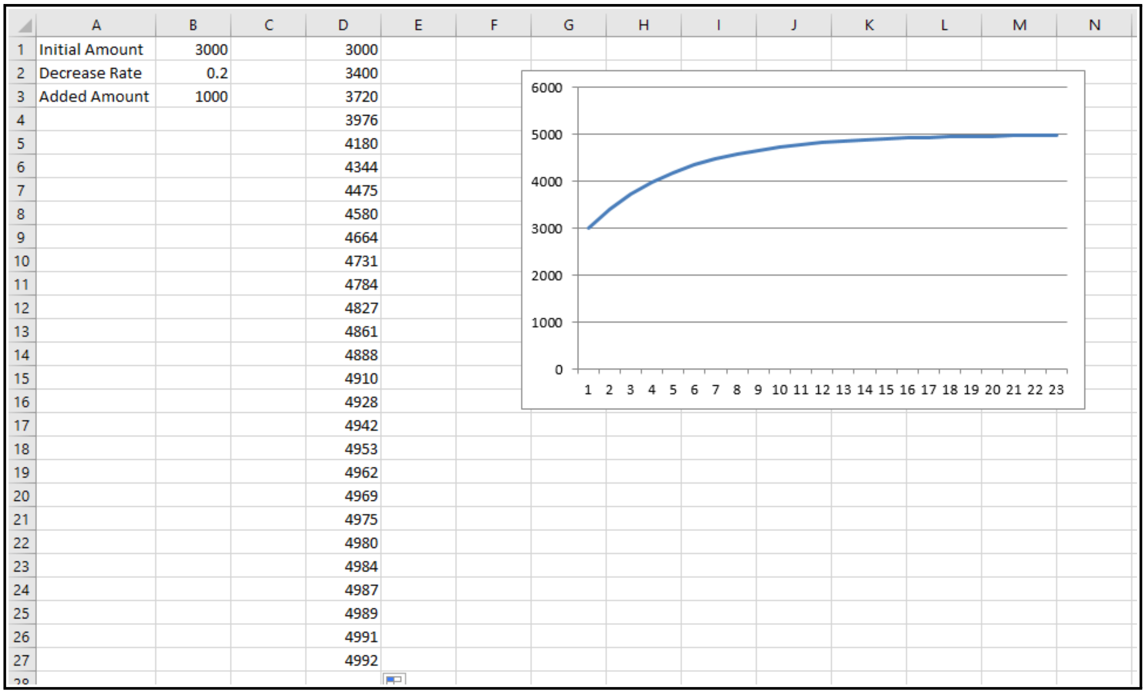
## 3 Revised Activity

Lannin (2004) and Lannin, Barker, and Townsend (2006) argue that the use of a spreadsheet can foster students' recursive reasoning. "The spreadsheet can shift the focus of instruction away from the traditional emphasis on procedures toward developing meaning for algebraic representations" (Lannin, Barker, and Townsend, 2006, p. 304). Moreover, the spreadsheet allows students to start with a numerical focus and move toward a bigger picture with graphical and recursive algebraic representation. Even though students use a calculator when filling out the table in Figure 2, their focus stays on the calculations and numbers. One way to set up a spreadsheet for this activity is to put 3000 in the first cell of a column (say `A1`) and write "`=0.8*A1+1000`" in cell `A2`. Next, students drag cell `A2` down to populate the column for as many years as they wish. However, we prefer to set up the spreadsheet by entering the information provided in the problem (see Figure 3). This setup allow us to ask "what if" questions later in the activity (see below).

| | A | B |
|---|---|---|
| 1 | Initial Amount | 3000 |
| 2 | Decrease Rate | 0.2 |
| 3 | Added Amount | 1000 |

**Fig. 3:** *Initial setup.*

Then we ask students how they could use this information in these three cells to fill out their table in Figure 2. First step is to copy the initial amount from cell `B1` which is done by typing "`=B1`" into any cell (in our case cell `D1`). Defining the cell `D2` as "`=INT(D1*(1-$B$2)+$B$3)`" is at the heart of this task. Let's unpack this formula. `D1` is the initial value and is multiplied with `(1-B2)`. The expression `(1-B2)` represents the remaining 80% of the fish in the pond. The dollar signs anchors the cell `B2` in this formula, since we will copy and paste down the formula in order to calculate the other years. Addition of the cell `B3` represents the fish added each year. Similarly, adding dollar signs before column and row references ensures that `B3` will be used in each year's fish calculation (rather than `B4`, `B5`, and so on). Finally, the "`INT`" ensures that we have an integer number every year. When we copy down the cell to `D3`, it will read as "`=INT(D2*(1-$B$2)+$B$3).`" As you will see, `D1` is placed with `D2` but the remaining formula is the same. The year 3 calculation (the cell `D3`) uses the second year (the cell `D2`) fish number in the pond. Adding a graph will help students see the trend more efficiently (see Figure 4). After seeing the numerical values and the graph, we ask students to consider *why the number levels off at 5000*. As students consider this question, they see that the restocking number and the decrease rate becomes equal at 5000. Another possible approach would be to drag down for just 10 years and ask students to reconsider their earlier conjecture. After seeing the population numbers for the first ten years, the apparent increase in population from year to year seems to eliminate students' popular "dying off" and "oscillation" conjectures.



| | A | B | D |
|---|---|---|---|
| 1 | Initial Amount | 3000 | 3000 |
| 2 | Decrease Rate | 0.2 | 3400 |
| 3 | Added Amount | 1000 | 3720 |
| 4 | | | 3976 |
| 5 | | | 4180 |
| 6 | | | 4344 |
| 7 | | | 4475 |
| 8 | | | 4580 |
| 9 | | | 4664 |
| 10 | | | 4731 |
| 11 | | | 4784 |
| 12 | | | 4827 |
| 13 | | | 4861 |
| 14 | | | 4888 |
| 15 | | | 4910 |
| 16 | | | 4928 |
| 17 | | | 4942 |
| 18 | | | 4953 |
| 19 | | | 4962 |
| 20 | | | 4969 |
| 21 | | | 4975 |
| 22 | | | 4980 |
| 23 | | | 4984 |
| 24 | | | 4987 |
| 25 | | | 4989 |
| 26 | | | 4991 |
| 27 | | | 4992 |

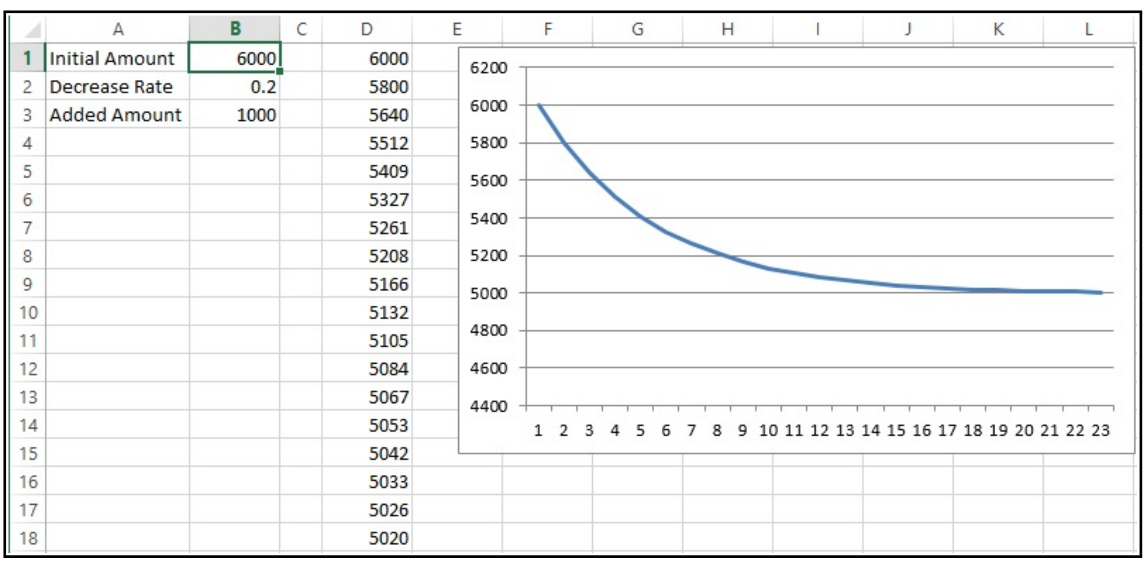


**Fig. 4:** *Completed spreadsheet.*

As mentioned above the advantage of this setup of the spreadsheet, is the ease with which we can now ask the following "what if" questions:

- What happens if we double the initial amount? (See Figure 5)
- What happens if we triple the restocking amount? (See Figure 6)
- What happens if the fish population decreases by 10%, 15%, etc.? (See Figure 7)

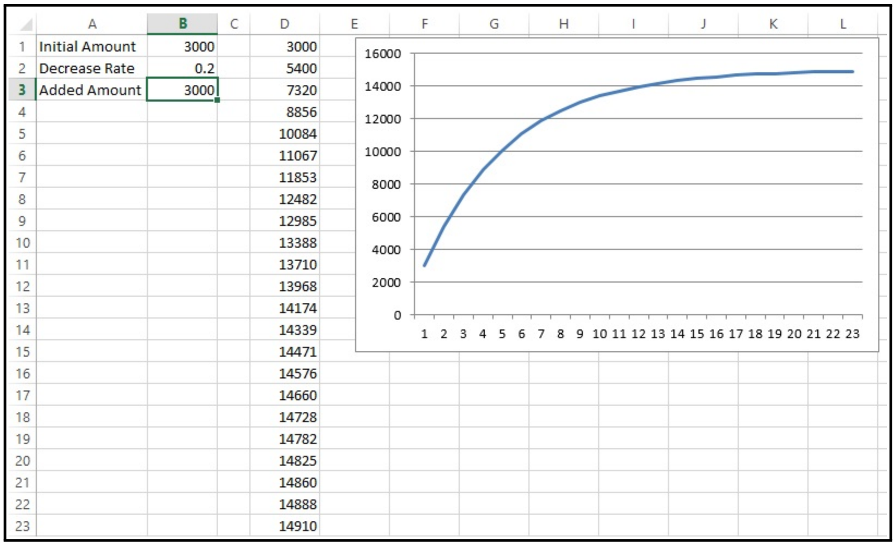
In order to answer these, students can change the values in B1 for initial amount, B2 for decrease rate, and B3 for restocking amount.



| | A | B | D |
|---|---|---|---|
| 1 | Initial Amount | 6000 | 6000 |
| 2 | Decrease Rate | 0.2 | 5800 |
| 3 | Added Amount | 1000 | 5640 |
| 4 | | | 5512 |
| 5 | | | 5409 |
| 6 | | | 5327 |
| 7 | | | 5261 |
| 8 | | | 5208 |
| 9 | | | 5166 |
| 10 | | | 5132 |
| 11 | | | 5105 |
| 12 | | | 5084 |
| 13 | | | 5067 |
| 14 | | | 5053 |
| 15 | | | 5042 |
| 16 | | | 5033 |
| 17 | | | 5026 |
| 18 | | | 5020 |

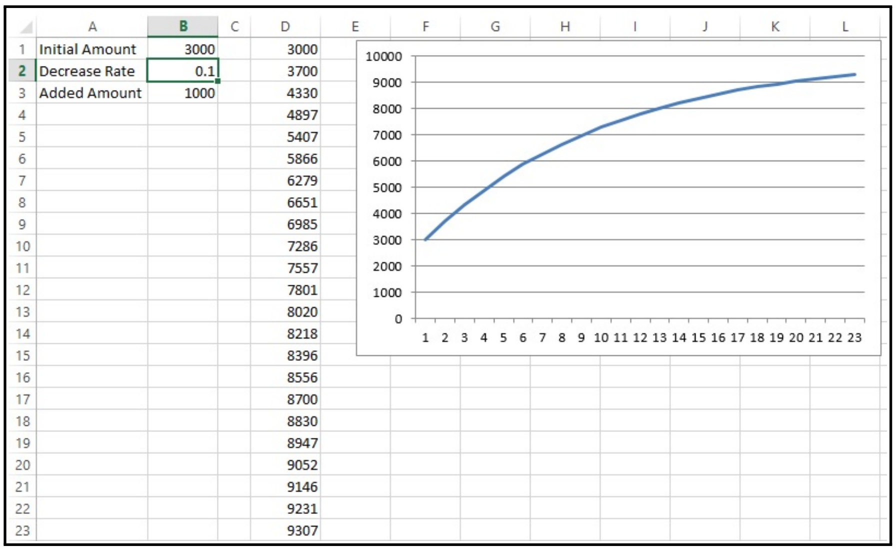


**Fig. 5:** *Spreadsheet with initial amount doubled.*



| | A | B | C | D |
|---|---|---|---|---|
| 1 | Initial Amount | 3000 | | 3000 |
| 2 | Decrease Rate | 0.2 | | 5400 |
| 3 | Added Amount | 3000 | | 7320 |
| 4 | | | | 8856 |
| 5 | | | | 10084 |
| 6 | | | | 11067 |
| 7 | | | | 11853 |
| 8 | | | | 12482 |
| 9 | | | | 12985 |
| 10 | | | | 13388 |
| 11 | | | | 13710 |
| 12 | | | | 13968 |
| 13 | | | | 14174 |
| 14 | | | | 14339 |
| 15 | | | | 14471 |
| 16 | | | | 14576 |
| 17 | | | | 14660 |
| 18 | | | | 14728 |
| 19 | | | | 14782 |
| 20 | | | | 14825 |
| 21 | | | | 14860 |
| 22 | | | | 14888 |
| 23 | | | | 14910 |

**Fig. 6:** *Spreadsheet with added amount tripled.*



| | A | B | C | D |
|---|---|---|---|---|
| 1 | Initial Amount | 3000 | | 3000 |
| 2 | Decrease Rate | 0.1 | | 3700 |
| 3 | Added Amount | 1000 | | 4330 |
| 4 | | | | 4897 |
| 5 | | | | 5407 |
| 6 | | | | 5866 |
| 7 | | | | 6279 |
| 8 | | | | 6651 |
| 9 | | | | 6985 |
| 10 | | | | 7286 |
| 11 | | | | 7557 |
| 12 | | | | 7801 |
| 13 | | | | 8020 |
| 14 | | | | 8218 |
| 15 | | | | 8396 |
| 16 | | | | 8556 |
| 17 | | | | 8700 |
| 18 | | | | 8830 |
| 19 | | | | 8947 |
| 20 | | | | 9052 |
| 21 | | | | 9146 |
| 22 | | | | 9231 |
| 23 | | | | 9307 |

**Fig. 7:** *Spreadsheet with a decrease rate of 10%.*

Many students are surprised to see that the initial amount does not change the "level-off number" although it can change the direction from which that number is approached. Further exploration allows students to see that the decay rate and replenishment amount can affect the "level-off number." Many observe the population approaches the quotient of the restock amount and the rate of decrease. Another class discussion might have students consider why it takes longer to reach the level-off number — namely, 10,000 — when the fish population decreases by 10%. Alternately, students can engage in a more open exploration and simply be asked to examine the effect of each of the "Initial amount," "Decay rate," and "Replenishment amount" on the population. In a similar fashion, one might ask students to examine the effect of changing the parameters $m$ and $b$ on the line $y = mx + b$.

In either case, it is important to ask students make guesses and share their reasoning behind their guess before they change the values in `B1`, `B2`, and `B3`. The use of advanced digital technologies can make it too easy to change values and press buttons without considering the effect beforehand.

## 4 Concluding Thoughts

The use of patterns to build explicit algebraic formulae is commonplace in the algebra curriculum. The use of recursive reasoning is less common perhaps, in large part, because of the considerable amount of calculation involved. Furthermore, problems involving population growth, for which explicit formulae are often beyond the level of many middle school or high school classes, can be particularly interesting for exploration through recursive thinking. The use of advanced digital technologies, in this case a spreadsheet, can relieve the burden of calculation and allow students to explore with a focus on the underlying concepts. However, management of the use of technology is vital so that students do not "mindlessly" press or click buttons without making predictions based on evidence and then use the technology to learn.

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