An Alternative Proof that the Real Numbers are Uncountable

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Abstract

Many students as well as educators find it difficult to accept Cantor's Diagonalization argument that the real numbers are uncountable. In this paper, we provide an alternative proof that may be easier for students to accept. The proof could provide further insight into the reason that the real numbers are uncountable. That is, any statement that leads to anything being possible includes the fact that the statement is impossible.

Keywords: Real numbers, Uncountability, Proof and Argumentation

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Background

Cantor's Diagonalization method has long been a cornerstone in demonstrating the uncountability of the real numbers (Cantor, 1891). While this technique is well-established, the present paper focuses on an alternative approach using Russell's Paradox, which may be more accessible to students and educators.

Without delving into the details, we acknowledge that students can have difficulties with Cantor's Diagonal argument (see Appendix A). While alternative proofs exist for the uncountability of the reals (Knaap and Silva, 2014; Gascón, 2015), they are often outside the reach of high school or undergraduate students. This paper presents a proof rooted in Russell's Paradox, specifically designed for accessibility. (Editors' note: For those less familiar with the notion of countability and uncountability, *Number Cruncher* provides an excellent overview at https://youtu.be/JmXDcAVR8k8?si=hXsY1jM8bB66kXCa.)

Our argument, in contrast, uses self-referential contradictions and is captured in the opening statement of this paper, "Any statement that leads to anything being possible includes the fact that the statement is impossible." To understand this proof, students should first grasp self-referential contradictions, such as the Liar's Paradox.

The Liar's Paradox

"I am lying to you now." If I am lying, then I am telling the truth. But if I am telling the truth, then I am lying ...

This self-referential contradiction cannot be determined to be either true or false. A similar self-referential contradiction lies at the heart of a paradox which will be used in our proof: Russell's Paradox.

Russell's Paradox

Discovered by renowned logician and philosopher Bertrand Russell, Russell's Paradox exposes a logical impossibility in set theory and logic more broadly. Let's consider a simple example: a library catalog listing all books not cataloging themselves. If such a catalog includes itself, it contradicts its premise because it would then be cataloging itself, which is against the rule. Conversely, if it excludes itself, it fails to follow its own rule of listing all books that do not catalog themselves, and thus it must be included. This paradox, applied to sets, highlights the inconsistency in assuming a complete list of real numbers. Specifically, Russell's paradox considers the set of all sets that are not members of themselves. But would such a set be a member of itself? If it is, then it isn't. If it isn't, then it is. In other words, this set "is neither a member of itself nor not a member of itself" (Weinstein, n.d.). Students who are being taught the contradiction that results from Cantor's Diagonalization argument also have the ability to understand our proof, which is shown in the next section.

Alternative Proof that the Real Numbers are Not Countable

Let us assume, for the sake of later contradiction, that the real numbers between 0 and 1 (inclusive) are countable. Since "...the continuum of numbers, or real numbers system ...is the totality of infinite decimals," (Courant, Robbins & Steward, 1996) then a listing of the real numbers between 0 and 1 in base 3 (a ternary numeral system) would have all possible sequences of the digits 0, 1, and 2 after the radix point[1] Now, let's examine our hypothetical complete list of base-3 numbers (the following table shows six examples from an infinite list):

Base 3 List (Hypothetical case)

1.	0.1210221101211001...	4.	0.0111111221002111...	
2.	0.1122222222222222...	5.	0.2222222222222222...	**etc.**
3.	0.1111111111111111...	6.	0.1011122211111100...	

Transforming Base 3 Numbers to a List of Base 2 (Binary) Numbers

For each of the numbers shown in this infinite list, we perform the following transformations:

(a) Remove the leading "0." from each number
(b) Replace each "2" with a space (" ") to separate sequences of 0s and 1s
(c) Interpret the resulting sequences between spaces as binary numbers
(d) Create a set consisting of the remaining numbers (a set of Base 2 (binary) numbers)

For example, consider the first Base 3 number in our list, 0.1210221101211001.... We implement each of the four steps below to transform the number into a set of binary values.

(a)	Start with:	0.1210221101211001...
(b)	Remove leading zero and radix ("0."):	1210221101211001...
(c)	Replace each "2" with a space (" "):	1 10 1101 11001...
(d)	Result:	{1, 10, 1101, 11001...}

[1] Some base-3 real numbers have multiple valid representations. For example, $0.1_3 = 0.022\overline{2}_3$, similar to how $1 = 0.999\overline{9}_{10}$ in decimal notation. This means some numbers in the list appear in more than one form, particularly those ending in all 2s. However, we can always ensure a unique list by systematically eliminating such duplicates. This does not impact the proof, as every 2 is later replaced in the transformation process, ensuring that the final set consists only of base-2 numbers (i.e., those with digits 0 and 1). The process of making the list unique can always be applied row by row without changing the structure of the argument.

The following table shows the resulting Base-2 (Binary) list generated by the transformation.

Transformed List (Base 2)

1.	{1, 10, 1101, 11001…}	4.	{111111, 100, 11…}	
2.	{11}	5.	{}	**etc.**
3.	{11111111111111…}	6.	{10111, 11111100…}	

Now, let's carefully examine the relationship between row numbers and their sets:

Each row number can be written in binary. For example:

- $1_{10} = 1_2$
- $4_{10} = 100_2$
- $6_{10} = 110_2$

Some rows cannot contain their own row number:

- Row 3 only contains an infinite string of 1s
- Row 5 is empty
- Any row containing only infinite sequences cannot contain its finite row number

Some rows might contain their row number:

- Row 1 contains 1
- Row 4 contains 100_2 (i.e., the Binary representation of the decimal number "4".)

A Crucial Observation

Since our original list was supposed to contain all possible real numbers between 0 and 1, our New List must contain all possible combinations of finite and infinite binary strings. Therefore, somewhere in our list, *there must be a row that contains exactly the binary numbers of all rows that don't contain their own row numbers.*

Here is where the similarity to Russell's Paradox emerges:

> Consider this special row that supposedly contains all row numbers that don't contain themselves. Should this row contain its own number?

Scenario 1: If the row does contain its own number, then by definition, it shouldn't be in the set (that contradicts our original assumption, and therefore can't hold.)

Scenario 2: If, on the other hand, the row doesn't contain its own number, then by definition, it should be in the set (this also contradicts our original assumption, and therefore can't hold.)

> This contradiction shows that our initial assumption—that the real numbers between 0 and 1 are countable—must be false. Therefore, the real numbers are uncountable.

We have shown a proof that the real numbers are uncountable using a self-referential contradiction similar in nature to Russell's Paradox. This is different than Cantor's Diagonalization argument, which is not a proof that relies upon Russell's paradox (Bell, 2004). In fact, chronologically, Cantor came up with his diagonalization before Russell came up with his paradox.

Final Remarks

This paper provides an accessible demonstration of the uncountability of real numbers that is reminiscent of Russell's Paradox, offering an alternative to Cantor's Diagonalization argument. Educators may find this approach particularly valuable for students who struggle with diagonalization, fostering deeper insights into the nature of infinity and uncountability. The reasoning reflects the paradoxical essence captured in the opening sentence of this paper: "Any statement that leads to anything being possible includes the fact that the statement is impossible."

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Appendix A

It may not be a surprise that some students have trouble accepting Cantor's Diagonalization argument. One paper in the literature, What is wrong with Cantor's Diagonal Argument (Brady and Rush, 2008), opens with the following:

> As a long-time university teacher of formal logic and philosophy of mathematics, the first author has come across a number of students over the years who have cast some doubt on the validity of Cantor's Diagonal Argument.

Although all proofs that the reals are uncountable may involve abstract or counterintuitive elements, Cantor's Diagonalization argument seems particularly challenging for students. It often makes the concept feel like a trick or even invalid.