Exploring Hyperbolic Geometry

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Abstract

The author describes an investigative approach to some basic concepts of hyperbolic geometry for high school students. As undergraduate students, most teachers have had a proof-based understanding of hyperbolic geometry. With the use of WebSketchpad, a free on-line dynamic geometry application, students and teachers can experiment with ideas from hyperbolic geometry that lay a foundation for the proofs.

Keywords: Hyperbolic geometry, Euclidean geometry, WebSketchpad, Interactive Geometry Software

1 Introduction

How well do you recall hyperbolic geometry from your undergraduate days? Hyperbolic geometry is one example of a non-Euclidean geometry where definitions, postulates and theorems are developed with a different parallel postulate than Euclid's. Hyperbolic geometry has been traditionally reserved for college level mathematics. The goal of this article is to propose an exploratory investigative approach to introduce high school students to an alternate form of geometry, namely hyperbolic geometry. Teachers can pick and choose which topics to explore. Content from the high school curriculum of Euclidean geometry is presented side-by-side with the same content from hyperbolic geometry, highlighting either the commonalities between the two geometries or the differences between. For example, the Triangle Inequality Theorem holds in both geometries, but the angle sum of a triangle is different between the geometries. Therefore, hyperbolic geometry can be used to reinforce the Euclidean concepts.

2 A Brief History of Hyperbolic Geometry

2.1 Euclid's Fifth Postulate

From a historical perspective, hyperbolic geometry was born from attempts to prove Euclid's Fifth Postulate. Since the publication of *The Elements*, mathematicians accepted Euclid's Fifth Postulate as true. The Fifth Postulate reads as

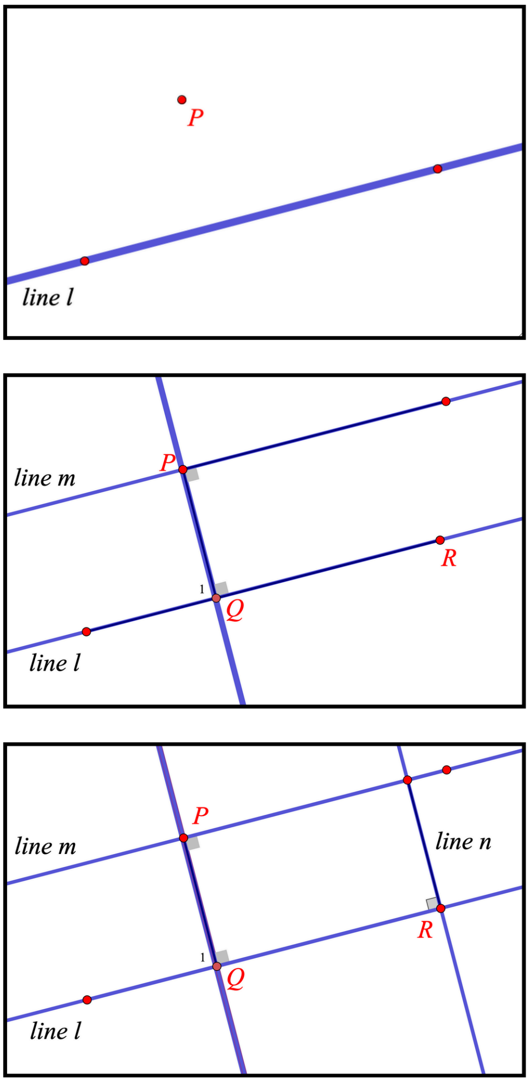
> That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles (Joyce, 1998).

More simply, the Fifth Postulate is stating that, given a line and an external point, there is exactly one line parallel through the external point to the given line. (The definition that two lines are parallel if and only if there is no intersection between the two lines will be used in this article.)

Historically, mathematicians were concerned about the validity of Euclid's Fifth Postulate from its publication. Among the earliest to contemplate its validity were Proclus, Ptolemy, and Aristotle. Aristotle's thinking of the Fifth Postulate, based upon the distance between parallel lines being constant, were lost until Arab scholars appeared to write upon Aristotle's treatment.[1] Khayyam criticized Euclid for his organization of the postulates.[2]

Mathematicians were curious if the Fifth Postulate could be proven using the previous four postulates. Gauss was among the first mathematicians to believe that the Fifth Postulate was independent of the other four, thus creating the opportunity for other parallel postulates to be logically possible. As a result, Bolyai, Lobachevsky and Riemann developed other geometries logically (Struik, p. 167). Bolyai and Lobachevsky developed hyperbolic geometry, where, given a line and an external point, there is more than one line parallel through the external point *P*. Consider the construction shown in Figure 1.

Figure 1: Steps to construct a dynamic rectangle.



Start with a line *l* and an external point *P*.

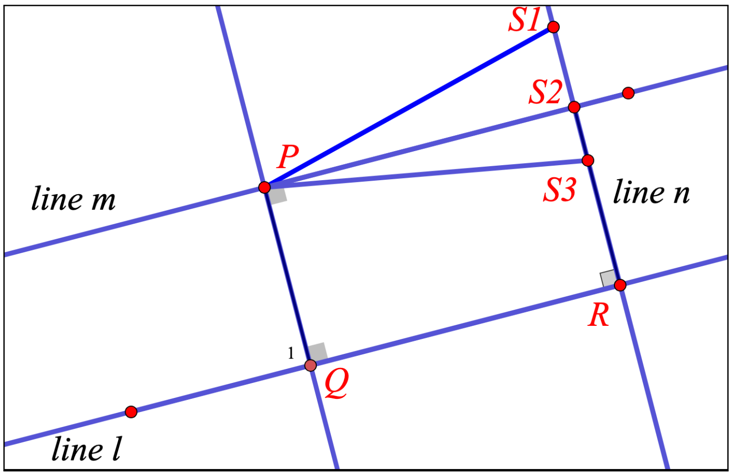
Construct a perpendicular from *P* to *l* with the foot point *Q*. Also construct a perpendicular to line *PQ* at *P* as line *m*. Line *m* is now parallel to line *l*.

Pick any point R on l and construct a perpendicular to l at R. We'll refer to that perpendicular line as n.

If we were to construct a perpendicular from *P* to *n*, where would its intersection, *S*, lie? There are three possibilities, denoted as S_1, S_2 and S_3 in Figure 2.

[1]Refer to https://tinyurl.com/Aristotle-Euclid.
[2]Refer to https://tinyurl.com/Khayyam-Euclid.

Figure 2: Where is the intersection?



After a year's study of Euclidean geometry, most of my students understand that the intersection at S_1 cannot occur because the angle sum of quadrilateral $PQRS_1$ would surpass 360°. The intersection at S_2 is what would occur in Euclidean geometry. For S_3, Euclidean geometry would not allow that intersection because the angle sum would now be less than 360°. But is there a geometry where it is possible? The answer is hyperbolic geometry. In hyperbolic geometry, $PS_3 || l$ since it has a double perpendicular with line n.

2.2 Hyperbolic Geometry

The focus of this article will be on transitioning from Euclidean to hyperbolic geometry through exploration. Hyperbolic geometry is closely related to Euclidean geometry because of the existence of parallel lines. As a first exposure to non-Euclidean geometries, a discussion of commonalities between the two geometries provides a smoother introduction. We follow a similar approach when exploring other geometries (e.g., elliptical or taxicab) with students.

For the activity discussed in this article, experimentation is the key to understanding. Experimentation does not constitute proof but does lead towards formulating conjectures. Students need to see numerous examples to best gain a foundational understanding of the information being studied. Many examples help to increase the chances of not getting that special situation where it looks like the conjecture is true but really is not (i.e., a counterexample). Remind the students that only one counterexample is necessary to prove a conjecture to be false.

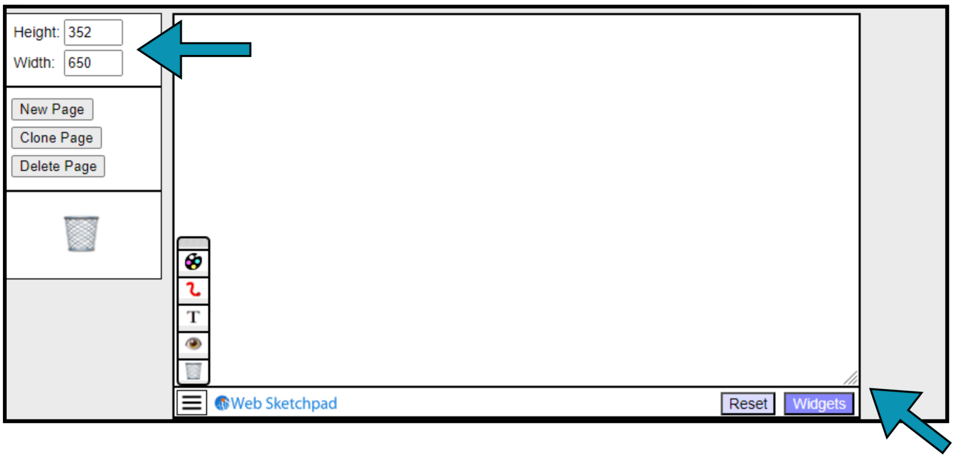
2.3 Introducing the WebSketchpad Tool Library

Proving theorems in hyperbolic geometry without experimentation will be beyond the grasp of most high school students. For those teachers who wish to have students create arguments based on hyperbolic geometry, it is vital that students experiment with the content first. One way to help students gain insight into the concepts of hyperbolic geometry is to use WebSketchpad (WSP), an online tool that provides students and teachers with the ability to explore geometric conjectures.

The WSP Tool Library is located at https://geometricfunctions.org/fc/tools/library/. (You may wish to refer to the WSP tutorials at https://geometricfunctions.org/fc/tools/ to see what WSP can do.) First, some basic instructions into WSP's use seems prudent.

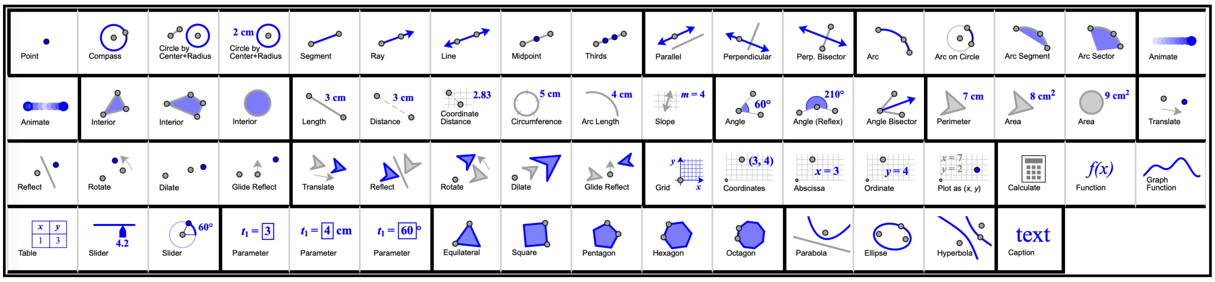
As Figure 3 suggests, the size of the page can be adjusted by either typing in directly the desired size of height and width or by dragging the diagonal lines in the bottom right corner of the page.

Figure 3: Changing the dimensions of the WebSketchpad interface.



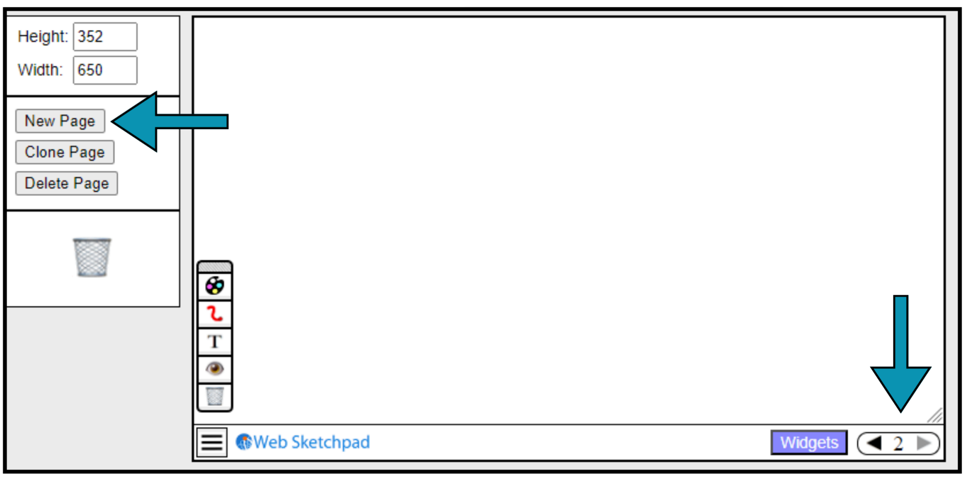
Tools, such as those illustrated in Figure 4, are provided at the bottom of the WSP interface. Tapping on a tool will make it appear in the page and then be available for use. There are two choices as to the tools. The Basic Tab contains not only construction and measurement tools required for Euclidean geometry but also general tools such as point, text and a calculator. Scrolling will show all available tools.

Figure 4: The WebSketchpad toolbar. Users click on a tool to add it to their sketch.



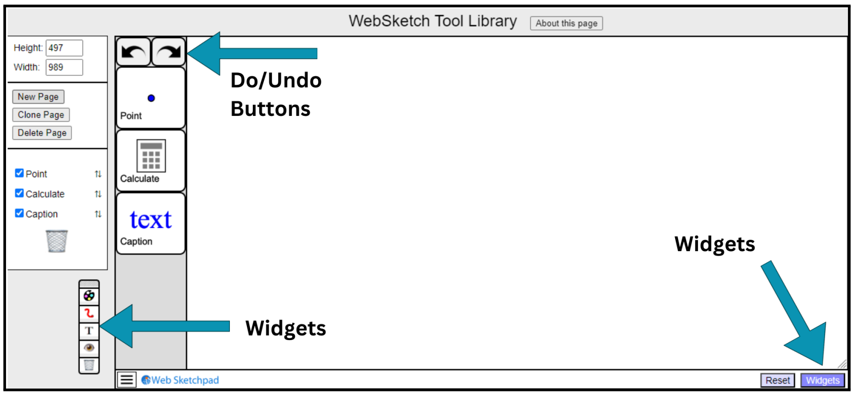
As Figure 5 suggests, clicking on New Page in the left column below the height and width of the window will create a new page within the same sketch. You may want a new sketch for each investigation. In the bottom right corner, the number 2 with arrows now appears. Clicking on the arrows will move you from page 1 to page 2 or back to page 1. Adding more new pages will create pages 3, 4 and so on.

Figure 5: The WebSketchpad navigation and New Page tools.



Size the page as you desire. Tap on the point, calculate and text tools under the Basic tab. These three tools should now appear on the page. The widgets may be in an inconvenient place; drag it by the top tab and move it. The widgets can appear or disappear by clicking on the widget button. If you make a mistake, use the undo and redo buttons (see Figure 6). As a tool is entered, it appears in the window below New Page. If that tool is no longer needed, drag its name in that window into the trash can.

Figure 6: The WebSketchpad undo/redo buttons and widget tools.

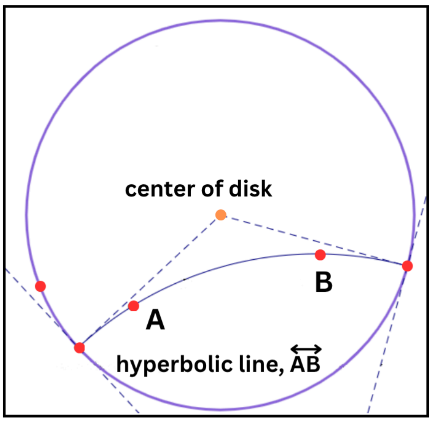


At any time while working on a sketch, you may wish to save your work. To do so, click on the three horizontal lines in the bottom left corner of the page. A pop-up menu appears; choose Download. Type on name that you wish but there shall be no spaces or characters in the sketch name, followed by the extension ".json." The sketch has now been saved in the Downloads folder of the computer on which you are working. To retrieve the sketch, open WSP, click on the same three horizontal lines and choose Upload.

For the purposes of this activity, choose the Hyperbolic geometry tab and tap on the Hyperbolic plane, segment, line, perpendicular and circle (CR) [center and radius] tools. These tools should now appear on the left side of the page.

The geometric model used by WSP is the Poincare disk instead of a Klein disk. In the Poincare disk, the points that exist are those that are in the interior of the circle, not the circle itself. A hyperbolic line must be perpendicular to the Euclidean tangent line at that point on the circle. In Figure 7, note how the "endpoints" of hyperbolic line $\overleftrightarrow{AB}$ are the points of tangency to the Euclidean tangent lines. This forces the hyperbolic lines and segments to appear curved.

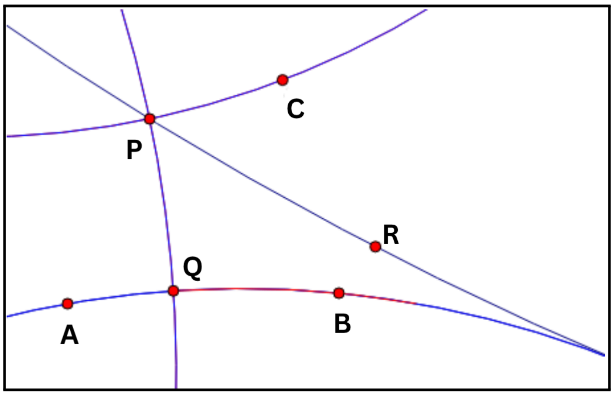
Figure 7: Poincare disk. The dashed lines represent Euclidean constructions of tangent lines.



The Poincare disk shows the multiple parallel lines easily. Bolyai's Construction finds the first parallel line other than the Euclidean parallel line. In Figure 8, $\overleftrightarrow{PQ}$ is perpendicular to both

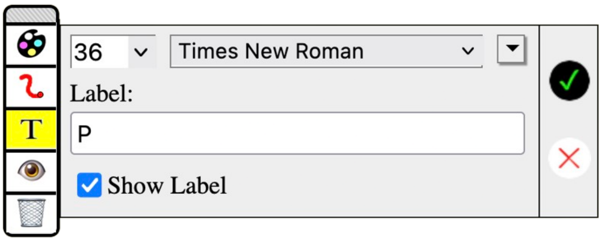
$\overleftrightarrow{AB}$ and $\overleftrightarrow{PC}$, making $\overleftrightarrow{AB}||\overleftrightarrow{PC}$. But it can also be shown that $\overrightarrow{PR}||\overrightarrow{QB}$, where $\overrightarrow{PR}$ is defined to be the limiting parallel ray. Any ray located between $\overrightarrow{PR}$ and $\overrightarrow{PC}$ will also be parallel to $\overrightarrow{QB}$; all rays between $\overrightarrow{PR}$ and $\overrightarrow{PQ}$ will intersect $\overleftrightarrow{AB}$.

Figure 8: Bolyai's Construction of the Limiting Parallel Ray.



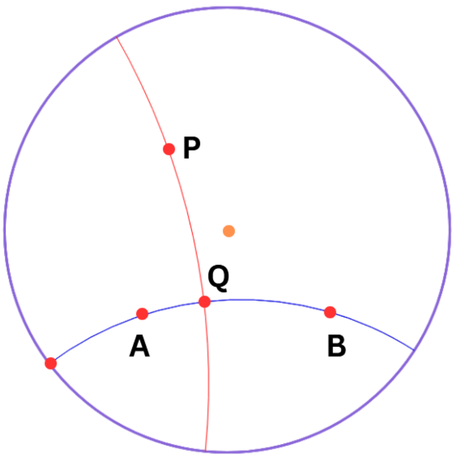
To make this construction, start by selecting the hyperbolic plane. Feel free to drag the center of the plane and the given point on the circle to size the plane to what you wish. You may wish to hide the disk controls so that you do not randomly change the size or location of the hyperbolic plane. Then add a line $\overleftrightarrow{AB}$ and an external point P. Depending upon the size of your hyperbolic plane, the point and line may appear outside of the circle. Drag them inside. To label points, choose the Label widget . Click the green check mark to finalize.

Figure 9: WebSketchpad's text editing dialog box.



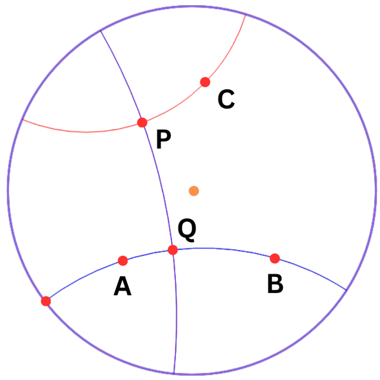
Now construct a perpendicular from P to the line by selecting the tool, then points A, B and P. Create and label the intersection as Q as shown in Figure 10. To do so, select the point tool and hover over the intersection until both lines turn red, then click.

Figure 10: Constructing perpendicular lines with WebSketchpad's hyperbolic tools.



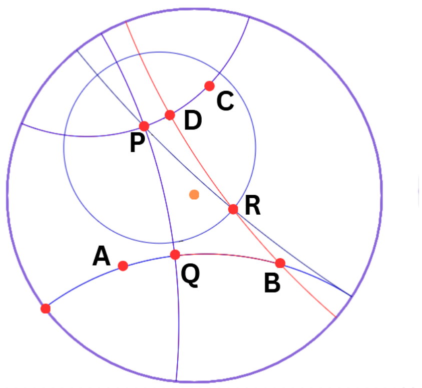
Now construct a perpendicular to $\overleftrightarrow{PQ}$ through P. Pick any point C on the newly constructed perpendicular.

Figure 11: Constructing a second perpendicular with WebSketchpad's hyperbolic tools.



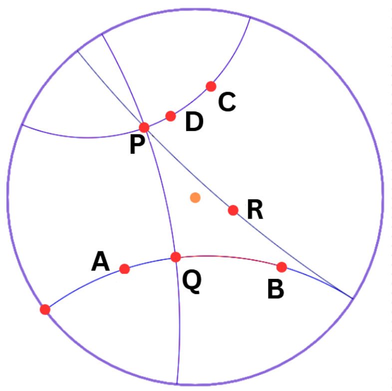
Now construct a perpendicular from B to $\overleftrightarrow{PC}$ with the intersection as point D. Lastly, construct a circle centered at P with a radius of $\overline{QB}$. Label the point of intersection between $\circ P$ and $\overline{BD}$ as R. Draw in the line $\overleftrightarrow{PR}$.

Figure 12: Constructing a triangle with WebSketchpad's hyperbolic tools.



You can hide any unnecessary objects by first selecting the Visibility widget which looks like an eyeball 👁. After its selection, click on any object that you do not wish to see any longer. Click on the widget to turn off hiding objects.

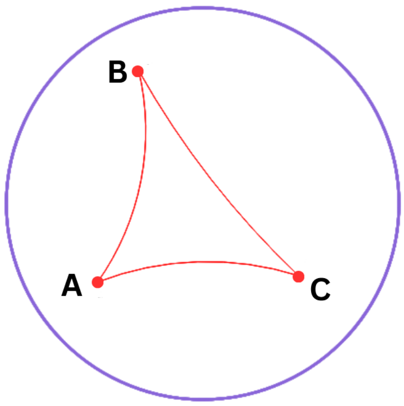
Figure 13: Constructing a triangle with WebSketchpad's hyperbolic tools.



Note the differences in the parallel lines $\overrightarrow{PC}$ and $\overrightarrow{PR}$ when compared to $\overrightarrow{QB}$. $\overrightarrow{PC}$ is part of the parallel line constructed with a double perpendicular; I call this the Euclidean parallel line. In Euclidean geometry, the distance between these two parallel rays remains constant. In contrast, in hyperbolic geometry, the distance between $\overrightarrow{PC}$ and $\overrightarrow{QB}$ increases. These rays would be called divergently parallel. In fact, $\overrightarrow{PQ}$ is the only common perpendicular between $\overrightarrow{PC}$ and $\overrightarrow{QB}$. We will investigate this shortly. The rays $\overrightarrow{PR}$ and $\overrightarrow{QB}$ appear to have a decreasing distance, thus said to be asymptotically parallel.

Let us revisit the idea of only one common perpendicular between parallel lines. To see this, let us find the angle sum of a triangle. In WSP, make a new page and choose the hyperbolic plane. Make any triangle you wish be joining the segments endpoint to endpoint. You will need the Measure Angle from the Hyperbolic tab. To measure an angle, click on the points as if you were naming the angle by three points. For example, to measure ∢B, select A, B then C. The measurements can be moved by clicking and dragging them by the equal sign to any location. To calculate the sum, first select the calculate tool, then the equal sign in the blinking box. This should create a dialog box entitled Edit Calculation. Click on one angle measure, the plus sign, another angle measure and the plus sign followed by the last angle measure and OK. (WSP may display the angle measures as related to the disk center. Ignore that.)

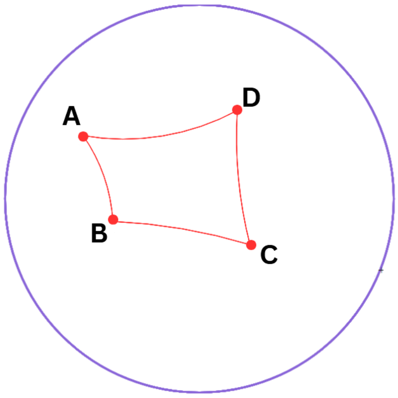
Figure 14: Constructing a triangle with WebSketchpad's hyperbolic tools.



Clearly, with the first example, the angle sum of a triangle is not equivalent to 180°. What is the lowest sum possible? What is the greatest sum possible? Drag points A, B and C. A pattern should develop, that of the larger the triangle, the smaller the sum. The angle sum of a small triangle may be displayed as a sum of 180° but it is a rounding error. (Note: The proof of this theorem is beyond the scope of this paper.) How often has the measure of the third angle of a triangle calculated by subtracting the other two measures from 180°? That is no longer available in hyperbolic geometry.

If the angle sum of a triangle is strictly less than 180°, what would the angle sum of a quadrilateral be? A student should conjecture that the sum would be strictly less than 360°; a quick sketch of a quadrilateral and its angle sum would help to confirm the conjecture.

Figure 15: Constructing a quadrilateral with WebSketchpad's hyperbolic tools.



Is it still true that a smaller quadrilateral yields a larger angle sum? How does this fact apply to how many double perpendiculars there are between two lines?

In Euclidean geometry, such a situation would result in a rectangle. But if the angle sum of a quadrilateral is strictly less than 360°, a rectangle would not exist in hyperbolic geometry.

3 Conclusion

In this article, we have introduced hyperbolic geometry in a historical context. Some of the differences between hyperbolic and Euclidean geometries have been explored, such as the number of parallel lines through an external point and the angle sum of a triangle and quadrilateral. In the next article, some of the effects of these facts upon angle relationships will be explored.

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