Keeping Modeling with Mathematics as the Focus in Authentic Tasks

Ryan Griesmer, Bowling Green State University
Stefanie D. Livers, Bowling Green State University
Thomas Roberts, Bowling Green State University

Abstract

Modeling with mathematics is essential in connecting mathematics to everyday life. In the preparation of teacher candidates, it becomes a mechanism to support the learning of mathematics as well as strengthening pedagogical approaches. This article focuses on a task that engaged teacher candidates in modeling with mathematics. After analyzing how teacher candidates solved problems, we categorized their methods into three categories of approaches. The alignment of the modeling task to the elementary mathematics curriculum is highlighted. Benefits of engaging teacher candidates in these types of tasks are also discussed.

Keywords: Mathematical modeling, teacher education, PK-5 mathematics

1 Introduction

The Standards for Mathematical Practice (SMPs), described in the Common Core State Standards, are standards that are framed around habits of mind that mathematics educators at all levels should strive to build in their students (National Governor's Association Center and Council of Chief State School Officers (NGA Center and CCSO), 2010). The fourth SMP is model with mathematics and is the purpose of the presented task. Characteristics that are associated with modeling with mathematics include solving problems that arise in everyday life (NGA Center and CCSSO, 2010). Students who are engaging in SMP four apply what they know to make assumptions and approximations to make sense of a given situation and create a mathematical model to communicate their solution and can explain the assumptions they made while developing their solution (Bostic et al., 2017). These types of tasks are important in connecting mathematics in the classroom to student's real experiences (Roberts et al., 2020). Modeling with mathematics should also be open which allows for multiple entry points for students (Greenhaus, 2016). This means that all students are able to engage with the task in a way that matches their current level of understanding.

Often around holidays, celebrations, and events, mathematics classes incorporate related activities. Unfortunately, these tasks are more focused on crafts and are actually non-mathematical activities (de Araujo, 2012), that do not include meaningful mathematics appropriate for a given grade level. In contrast, the presented task was designed to engage students in real world problem solving around a popular event, with a focus on modeling. In order to prepare for Halloween, planning and budgeting are necessary. It was presented to students as a Problem Statement (See Figure 1).

Figure 1: Problem Statement.

> Halloween is coming up, and you are responsible for buying the candy to pass out to the children in the neighborhood. In your planning, you look at the neighborhood map to determine how likely it is for specific groups to come by your house.
>
> In addition, you know how many children each of your neighbors have, and you also know their allergies. You need to decide how much of each bag of candy that you are going to buy to have enough candy for all the children in your neighborhood.

Students were also provided with the directions listed in Figure 2 and information about the neighborhood that a family lives, including their neighbors' number of children, dietary restrictions of the children, and how far away each house is from theirs (See Figure 3).

Figure 2: Directions for task.

1. Get Started. Familiarize yourself with the map and data table of the neighborhood (Editor's note: See Figure 3).

a. What are the possible factors that could impact the amount of and type of candy that you buy for the children?
b. Examine the map given. This should help you plan on deciding the groups of children that will visit your house.
c. Look at the number of children each of your neighbors have. Take allergies into consideration here.
d. Inspect the different bags of candy. How much candy does each bag have? How much does the bag cost?

2. Create your shopping list. Your group should ultimately come up with a list of the bags of candy that you will buy, along with the quantity of each bag.

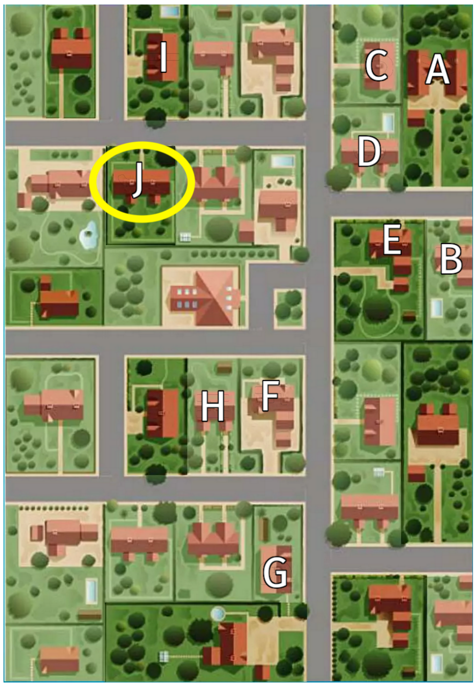
a. Your shopping list should also include a **mathematical model** to justify why you chose the types and quantity of candy bags that you chose. In this case, a mathematical model refers to a graph, equation, table, diagram, or formula that your group utilized to come to your decision.
b. Your group should also list any assumptions that you made throughout this process.
c. You want to be efficient with both money and candy. You should buy the fewest number of bags possible to maximize your budget and to reduce candy waste.

3. Share your list. Your group should upload your shopping list and your mathematical model (it can be in picture form!) to a discussion board. Be sure to explain your thinking in addition to showing your work with a picture, table, and/or numbers.

Figure 3 includes a simplified list for illustration purposes. The full list used in class included 30 houses to accommodate an entire class of students.

Figure 3: Neighborhood Information (Simplified for Illustration)

House	Children	Distance	Restrictions
A	3	0.8 mi	Nut allergy
B	3	0.9 mi	Vegan
C	1	0.7 mi	Diabetic
D	4	0.6 mi	Lactose intolerant, Nut allergy
E	5	0.6 mi	2 Nut allergies
F	0	0.6 mi	None
G	4	0.9 mi	Nut allergy
H	3	0.4 mi	Diabetic
I	2	0.2 mi	None
J	Your house	0 mi	None



As Figure 4 illustrates, students are also given the information about a variety of different bags of candy including quantity, cost, and dietary restrictions. Students use their knowledge about fractions, unit rates, and multiplication to create a model that describes how many and what type of bags of candy they would buy for a neighborhood Trick-Or-Treat.

Figure 4: Candy Options for Halloween

Options	Types of Candy	Price	Quantity
Bag 1	Snickers (milk chocolate, peanut)	$10	45oz
	100 Grand (milk chocolate)		
	Peanut M&Ms (milk chocolate, peanuts)		
	Twix (milk chocolate)		
	Regular M&Ms (milk chocolate)		
	Milky Way (milk chocolate)		
	Reeses Peanut Butter Cups (milk chocolate, peanut butter)		
	York Peppermint Patties (milk chocolate)		
	Almond Joy (milk chocolate, coconut, almonds)		
Bag 2	Skittles	$8	46oz
	Jolly Rancher Chews		
	Mini Air Heads		
	Original Starbursts		
	Twizzlers		
	Sour Patch Twists		
	Warheads		
	Albanese Gummy Bears		
	Sweet Tart Ropes		
	Chupa Chups		
	Haribo Gummy Bears		
	Life Saver Gummies		
Bag 3	Fruit Snacks	$8	80 packages

This task was implemented with two different groups of sophomore Inclusive PK-5 Education teacher candidates at Bowling Green State University. Although we ran this task with teacher candidates, the content in this task lends itself to be implemented in third through eighth grade classrooms. Table 1 shows the alignment of this task with relevant content standards for Grades 3-8.

Table 1. *Alignment to Ohio's Learning Standards for Mathematics (Grades 3-8)*

Level	Standards	Modifications to Task	Connection to Standards
3rd	**3.NF.1** Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; **3.MD.3** Create scaled picture and/or bar graphs to represent data sets with several categories.	No adaptation needed. For third grade, the teacher may give one street instead of a neighborhood or reduce the number/type of dietary restrictions.	For **3.NF.1**, students may represent children with dietary restrictions as a fraction of the total children. For **3.MD.3**, students may create a scaled bar graph to show dietary restrictions or distances.
4th	**4.NF.3** Understand a fraction a/b with a > 1 as a sum of fractions 1/b; **4.MD.2a** Using models, add and subtract money and express answers in decimal notation; **4.MD.4** Display and/or interpret data in graphs to solve problems using numbers and operations.	For **4.NF.3**, include a visual fraction model. For **4.MD.2a**, provide a budget for spending, including adding/subtracting money. Teachers may give two or three streets instead of a whole neighborhood and reduce dietary restrictions.	For **4.NF.3**, students represent quantities as a sum of fractions. For **4.MD.2a**, students use models to represent money calculations. The task meets **4.MD.4** as students use graphs to display and/or interpret data.
5th	**5.NF.5a** Compare the size of a product to one factor based on the size of the other factor without performing the multiplication; **5.MD.2** Display and/or interpret data in graphs to solve problems using numbers and operations in fractions/decimals.	For **5.NF.5a**, include a variety of candy bag options. Teachers may give a full neighborhood but reduce the number of children or dietary restrictions.	For **5.NF.5a**, students think about the size differences in candy bags and their impact on purchasing decisions. The task meets **5.MD.2** as students use graphs to display and/or interpret data.
6th	**6.RP.1** Understand the concept of a ratio and use ratio language to describe a relationship between two quantities; **6.SP.4** Display numerical data in plots (line plots, histograms, box plots).	No adaptation needed. Teachers may provide more potential bags of candy for students to find ratios.	For **6.RP.1**, students use ratio language to describe relationships between candy bags and price per ounce. For **6.SP.4**, students use line plots, histograms, and dot plots to display data.
7th	**7.RP.3** Use proportional relationships to solve multistep ratio and percent problems; **7.SP.1a** Differentiate between a sample and a population.	For **7.RP.3**, every bag of candy is taxed. For **7.SP.1a**, add an extension where students discuss if their solution applies to everyone who buys Halloween candy.	For **7.RP.3**, students calculate tax to find the total spent. For **7.SP.1a**, students discuss sample versus population.
8th	**8.EE.5** Graph proportional relationships, interpreting the unit rate as the slope of the graph. Compare different proportional relationships represented in different ways.	Have students create a graph with the number of ounces in a single bag, the cost, and how it changes for buying multiple bags.	Students engage in **8.EE.5** by thinking about proportional relationships and scale factors, and how the unit rate impacts the graph.

2 Exploring the Mathematics

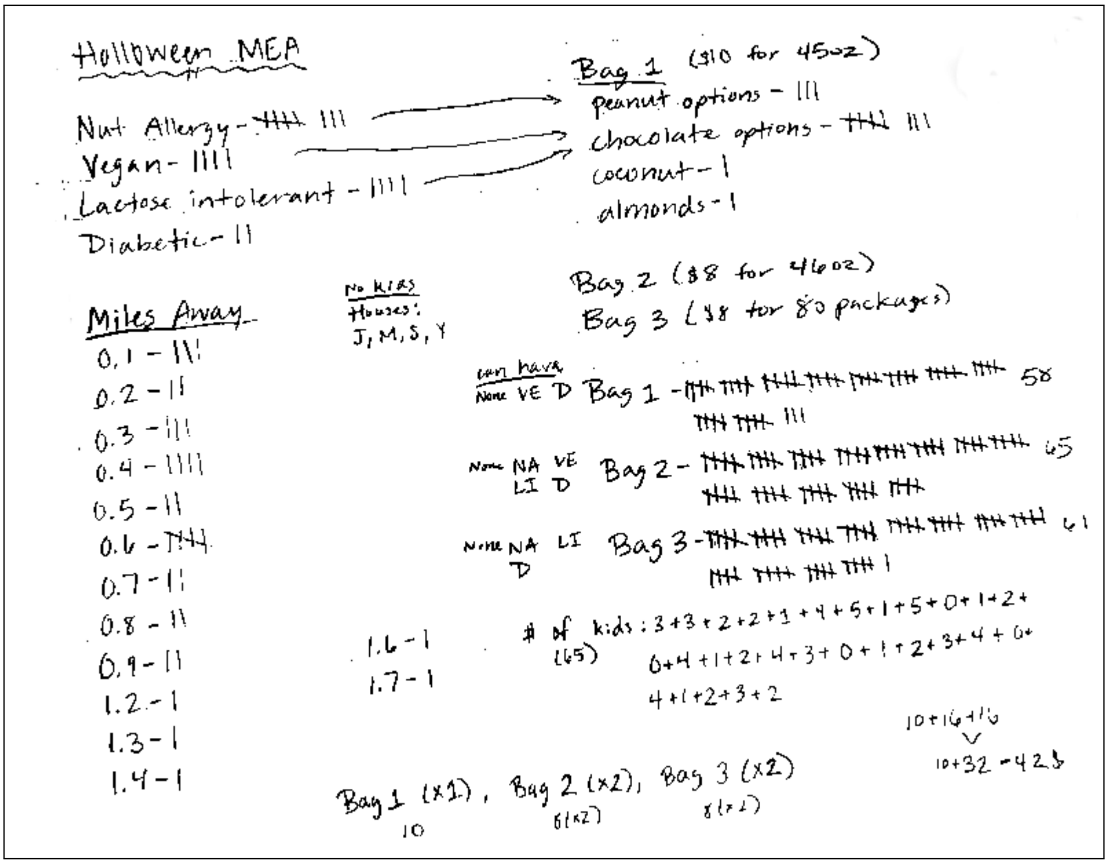
In addition to having multiple entry points, modeling with mathematics tasks allow for multiple exit points (Greenhaus, 2016). This means that the task allows for multiple possible student solutions that follow from the beginning entry points. Following the completion of this task, students shared their models, explanations, and final shopping lists to an online discussion board. The online discussion board was used for students to go through and look through their peers' responses. The discussion board responses were then analyzed according to how students modeled with mathematics (SMP Four) and coded into groups. We also took anecdotal notes during the class experience. Teacher candidate discussion board responses typically fell into one of three categories: solving the problem graphically, using scale factors to determine quantities, and using fractions to represent child populations. Exemplar responses are discussed below according to the three categories: solving the problem graphically; using scale factors to determine quantities; and utilizing graphs and fractions. In addition, questions posed during the task are provided, as well as student quotes and group discussion summaries.

Solving the Problem Graphically

Group 1 used a graphical approach to their problem (see Figure 5). This means that they worked through the problem by creating some type of graph to represent the data, and then used that graph to make conclusions about the problem. This group used a tally chart to organize their data. They started by identifying how many children had specific dietary restrictions. They listed out each dietary restriction given to them in the problem and how many children had that restriction. Then, they sorted the different types of candy in bag one based on specific dietary restrictions that it may cause, again using a tally chart. For example, bag one had three different candies with peanuts, which would not be good for the children with a nut allergy. After that, they listed out the different distances that each house was from their house. For example, they saw that it was given that five houses were 0.6 miles away from their house. They also identified which houses were childless. Next, they listed out each of the three bags and tallied how many children would be able to eat candy from that bag. They concluded that 58 children could eat from bag one, 65 could eat from bag two, and 61 could eat from bag three. These ideas are aligned to standard 4.MD.4, as students displayed their data of the distance between houses and their data for the number of children who can eat each bag through two different tally charts. Students then interpreted their data to conclude that because the majority of children could eat from bag two, they would buy two bags. Although bag one is the most restrictive, they still wanted to have some chocolate options, so they decided to get one bag that contained chocolate. Lastly, they wanted some low sugar options, so they also got two bags of bag three. This brought their total for Halloween candy to be $42.

While facilitating the task, one question that we asked this group is "What do each of your charts represent?" This helped the group think more deeply about the data that the group was working with and how that may change their overall solution. After a few seconds to think, one of the group members told us that "the 'Miles Away' chart represents the number of houses that are a given distance away from their house, in miles." The group members said that this was helpful to them so that they could track how many children may come to their house. They also discussed how the "Can Have" chart represents the number of children that can have candy from each bag. Another purposeful question that we asked this group was "Now that you know how many children can eat candy out of each bag, how will you determine the overall amount of candy that you need to buy?" Although this group did not have time to deeply think about this question, it could have brought up ideas related to proportions or scale factors.

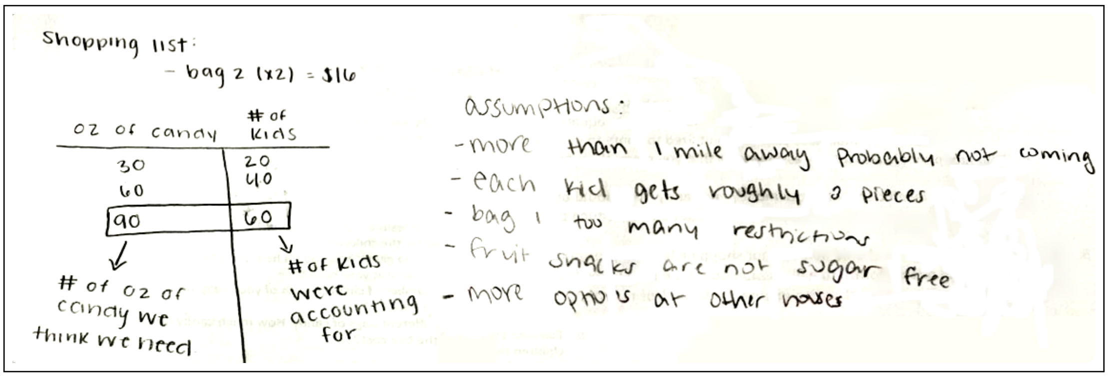
Figure 5: Graphical approach (Group 1).



Using Scale Factors to Determine Quantities

Figure 6 illustrates the work of Group 2. These students used a scale factor approach, which can be used to determine how much candy to buy for a large number of children. Students using this approach solve for a small quantity then multiply by a fixed scale factor.

Figure 6: Scale factor approach.



Group 2 started the task by removing every child that lived more than one mile away from the house. This was because they assumed that children who live more than one mile would be unlikely to walk over a mile to get to their house. Similarly to student group one, this group counted all of the allergies within the neighborhood, and looked at the contents of each bag to see who could and could not eat specific types of candy. They also concluded that the first bag had too many dietary restrictions, so they decided to not buy any of bag one. They created a chart to describe how many ounces of candy they need for a given number of kids. They decided that for 20 children, they would need 30 oz of candy. Then, using a scale factor of three to account for 60 children, they concluded that they needed 90 oz of candy. This approach focuses on standard 6.RP.1 as students found a ratio relationship between the ounces of candy and how many children they needed to pass out candy too. They then used that relationship to estimate how much candy they would need for every child in the neighborhood. This will provide enough candy for each child to have between two and three pieces. To reach the 90 oz of candy needed, they decided to buy two bags of bag two. This group also decided not to get any bags of bag three, meaning their total amount spent on Halloween candy was $16.

One question we posed to this group while they were working is “Is your strategy for determining how many ounces of candy you need generalizable for any number of children?” We gave the group some time to discuss and think about this question. When we came back to the group, one group member said, that “We believe that our strategy is generalizable, because no matter the number of children, you could just multiply that quantity by 1.5 and the product will be the ounces of candy that you need to buy.” This question allowed the students to think deeply about ratios and scale factors and how they can be used to find quantities.

Using Graphs and Fractions

Group 3 created their solution using graphs and fractions. Paper and pencil work is shown in Figure 7, while Figure 8 illustrates a pie chart that the group generated.

Figure 7: Fraction work (Group 3).

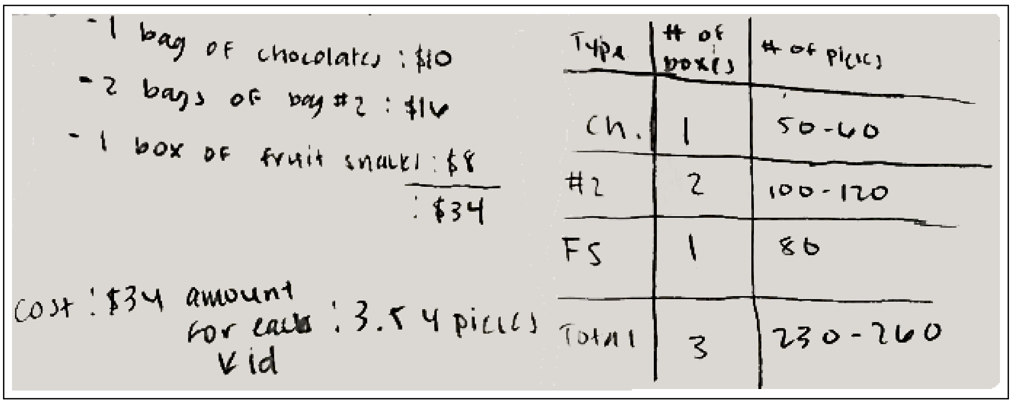
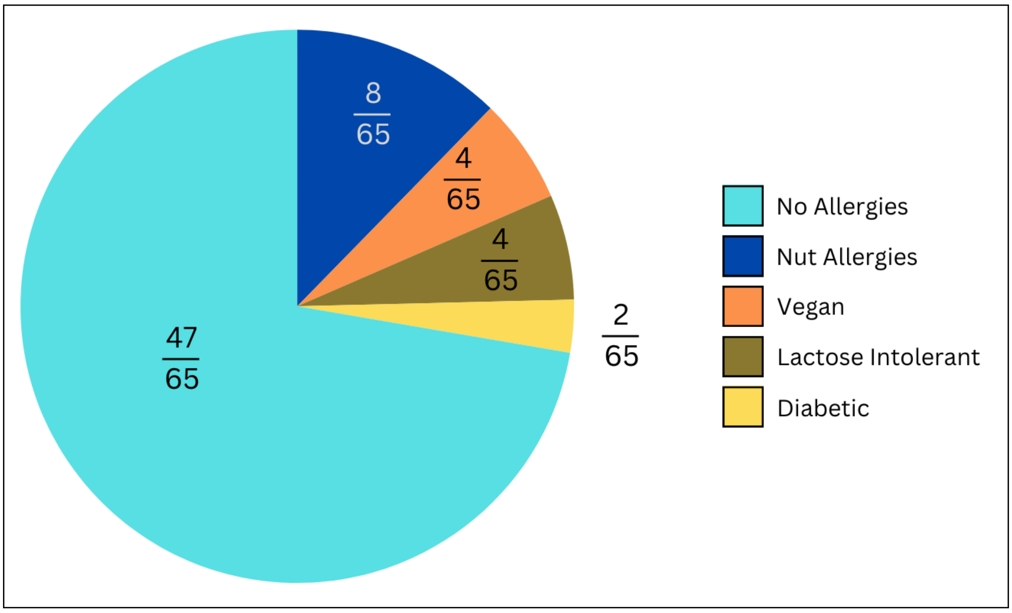


Figure 8: Distribution of allergies among kids in the neighborhood (Group 3).



The group described their process in the following manner:

> We started off by tallying up the number of kids total in the neighborhood and came up with 65 kids. We then tallied the number of allergies in the neighborhood which was a total of 18 out of the 65 kids. $\frac{8}{65}$ have nut allergies, $\frac{4}{65}$ are vegan, $\frac{4}{65}$ are lactose intolerance, and $\frac{2}{65}$ are diabetic, which leaves $\frac{47}{65}$ having no allergies. This is what our pie chart shows. We then go to see how much and what kind of candy we will need. We looked at the bags on the table and estimated how much would be in half of each bag. We estimated 50-60 pieces in each bag. Then obviously there would be 80 fruit snack packets in the box. We ultimately decided to have one box of the chocolate candies, this would satisfy the kids without the allergies, two boxes of bag #2, this would satisfy the children with allergies besides diabetics and would also be enough for children without allergies who wanted them to have some, and one box of the fruit snacks specifically to satisfy the diabetics, but also enough for all children to have some. There would be a total of 230-260 pieces of candy giving 3-4 pieces to each child. This would leave us to the only thing we really have to consider is watching which candies we give to what children.

The graphical approach is consistent with what was said earlier, as this group created a pie chart to represent the data that they had. The pie chart also represented different fractions that the group came up with. Fractions can be used to represent a variety of mathematical concepts, but this group chose to use them to represent the amount of children with special dietary needs. This group started by finding the total number of children in the neighborhood which was 65 children. Then, they found the number of allergies out of all of the children and found that 18 of the 65 students have a dietary restriction.

Utilizing fractions, the group represented each of these amounts as a fraction. This approach follows standard 4.NF.3 as students subtracted these fractions from the whole to determine the amount of children who did not have any dietary restrictions. Once they had a fraction for each of these quantities, they used a graphical approach by creating the above pie chart to represent the number of children that fall into each of the groups. In the pie chart, you can see that the fractions add to $\frac{65}{65}$, or one. Using bags of candy, the student group estimated that there are roughly 50-60 pieces of candy in bag one and bag two.

From there, the group decided to get one bag of bag one, two bags of bag two, and one bag of bag three. They thought that bag 1 would satisfy children who wanted chocolate candy. Then, bag two would be the primary candy that they hand out to most children. They also decided to buy one bag of fruit snacks to ensure that diabetic students had something they could eat, along with giving all children fruit snacks. They then used a table to determine the total cost of their candy, which was $34. They then divided the total number of candy by the total number of children in the neighborhood, to get that each child will receive 3.54 pieces of candy.

While observing this group's work on the task, one question we asked to further student thinking was "What do each of the sections of your pie chart represent?" The group discussed how each section of the chart represents the percentage of children that fell under each of the categories. We followed this by asking "What other representational strategies could you use to show these quantities?" This is when the group realized that in addition to their pie chart, they could represent the amounts using different fractions that sum to one.

3 Discussion

Tasks that engage students in SMP four, *Modeling with Mathematics*, are beneficial for a variety of reasons. One reason why these tasks are beneficial for students is that it engages students in mathematics while connecting their lived experiences to mathematics that they are learning in their classroom (Bostic, 2015). In the above task, teacher candidates were able to connect their knowledge about fractions, graphs, proportions, and multiplication to their experiences planning for Trick-or-Treating. These tasks are also beneficial for teacher candidates and elementary and middle school students because they provide an opportunity to reason with known information and create assumptions when completing a task (Chamberlin et al., 2020). This resembles the process that group two did in Figure 6 when they listed out all of their assumptions for the task. There are many different solutions that students could come up with for these types of tasks; the key within each solution is that they are able to explain their mathematical model and stated assumptions to show why their solution is a viable solution for the given task (Wolf & Ray-Riek, 2015), like Group 3 did in Figure 7. These types of tasks provided students with the opportunity to make connections between numbers and other structures that they may not have previously considered (Chamberlin et al., 2020), like Group 1 did in Figure 5 with how the data is represented in a tally chart.

The task above also showed an example of a modeling with mathematics task that was accessible to all students, with multiple entry and exit points. Designing the task to be open-ended provided each group to use their mathematical knowledge to begin thinking through a possible solution. Once each group had their entry point into the task, they were able to work through a solution pathway to solve the problem, come up with a unique solution, and present their solution to the class. Overall, modeling with mathematics tasks should provide all students with the opportunity to use their lived experiences and previous mathematics knowledge to begin working on a task, develop a solution pathway, and come to a justifiable solution.

We recognize the limitations of a task designed around an event that is not celebrated by all students. Although this task is focused on Halloween and Trick-Or-Treat, it could easily be modified to be inclusive for all learners. For example, in March, it is common for Jewish people to go door to door in their neighborhood to celebrate Passover. This task could be adapted to be relevant to that specific holiday. It could also be adapted to celebrate Christmas, Hanukkah, or Kwanzaa where people are trying to receive gifts from other people in their neighborhood. Additionally, it could be adapted to a school or community function. A quality task can be modified to fit the characteristics of diverse classrooms.

If we want teachers to teach through modeling, it is important to support teacher candidates in modeling with mathematics tasks. Teacher candidates need to experience the teaching and learning of mathematics through tasks in order to reframe and reconceptualize the teaching and learning of mathematics in order to move away from direct teacher centered approaches (Bezuk et al., 2017; AMTE Standards pp. 2–3). Carefully planned methods within teacher preparation can change traditional beliefs about mathematics instruction (Swars et al., 2009). We recommend that mathematics educators provide ample time to explore high-quality tasks throughout a semester. It is important for mathematics teacher educators to “create opportunities for learning subject matter that would enable teachers not only to know, but to learn to use what they know in the varied contexts of practice” (Ball & Bass, 2000, p. 99). Additionally, teacher candidates need to work collaboratively to see the value in allowing children to work together to process, plan, and work to solve problems. We also recommend doing the tasks with teacher candidates and then having them do the tasks with elementary and middle school children. Focusing on modeling with high-quality tasks allows teacher candidates and children to view mathematics differently by highlighting the various strategies that can be used to solve complex problems. Modeling based teacher preparation assists teacher candidates in developing perspectives that value real world contexts and see them as vehicles to teach mathematical concepts and opportunities to reason (Sevinc & Lesh, 2018).

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Ryan Griesmer graduated from Bowling Green State University in 2023 with a degree in secondary mathematics education. Ryan is passionate about fostering positivity and advancing education through innovative teaching methods. In addition to his academic interests, he is an avid runner who believes in maintaining a balanced and active lifestyle. His dedication to education and personal growth reflects his commitment to inspiring the next generation of learners.

Stefanie D. Livers is an Assistant Professor of Inclusive PK-5 STEAM in the School of Inclusive Teach- er Education at Bowling Green State University. A former elementary teacher and instructional coach, Dr. Livers' research agenda focuses on teacher preparation, teacher support, and equitable teacher practices. She is a National Board Certified Teacher who continues to work with schools and districts implementing cognitively demanding tasks and fostering equitable and just mathematical practices.

Thomas Roberts is an Associate Professor and PK-5 Co-Coordinator at the School of Inclusive Teacher Education at Bowling Green State University. Dr. Roberts' research interests focus on STEM education, informal learning environments, and equity in educational outcomes. His work aims to bridge the gap between theory and practice, fostering inclusive and effective educational practices.