Gauss's Area Formula for Irregular Shapes

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Abstract

To find the area of an irregular shape, we break the shape into common shapes. Then we find the area of each shape and add them, but this approach does not always work. In this paper we investigate Gauss's area formula (for irregular shapes), also known as the shoelace formula or the shoelace algorithm. This theorem is outside the scope of school program. Nevertheless, we provide several applications that emphasize the importance and usefulness of this theorem. Most of the applications provided in this paper are created by the authors.

Keywords: Shoelace theorem, area formulas, irregular polygons, alternative algorithms

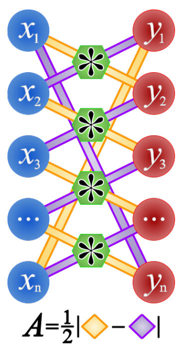
1 Introduction

This formula was described by a Prussian (German) mathematician *Albrecht Meister* in 1769 and is based on the *trapezoid formula* which was described by a prominent Prussian (German) mathematician *Carl Gauss* and *Carl Jacobi*.

$$A = \frac{1}{2}\Big|\Big(x_1y_2 + x_2y_3 + \ldots + x_ny_1\Big) - \Big(y_1x_2 + y_2x_3 + \ldots + y_nx_1\Big)\Big|.$$

It is known as *Gauss's area formula* (for irregular shapes), it is also known as *the shoelace formula* or *the shoelace algorithm* for the following reason: if we list x_1, x_2,..., x_n on one side and y_1, y_2,..., y_n on the other side and connect them by line segments, then we get a shape that resembles a shoelace as shown in Figure 1.

Figure 1: Graphical representation of shoelace algorithm.



The formula is especially useful because summing other shapes together is often tedious and unwieldy, and works poorly when the shape has very many vertices to consider, or when the positions of one or more of the shapes vary.

2 Using the Formula

To use Gauss' area formula for irregular shapes (the shoelace formula), first list all of the vertices of the shape whose area is being computed in the order that they occur (i.e., write the first point, then the point connected to it, and the point connected to that, and so forth).

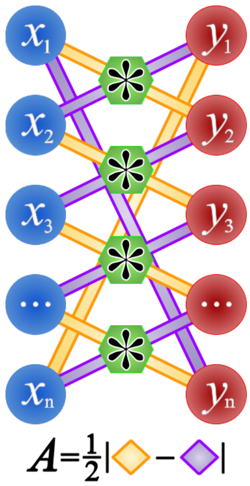
It helps if they are listed in the shoelace formation suggested in Figure 1.

Next, multiply the x-value of the first point by the y-value of the second point, then multiply the x-value of the second point with the y-value of the third, and so on and so forth until finally the x-value of the last point is multiplied by the y-value of the first point. Sum all of these products together.

Repeat this process, only this time multiply the y-value of the first point by the x-value of the second and so on. Sum those products together as well.

Subtract the second sum from the first sum. The area of the shape is half of the absolute value of that number.

Figure 2: Enlarged shoelace formula graphic for greater ease of access.

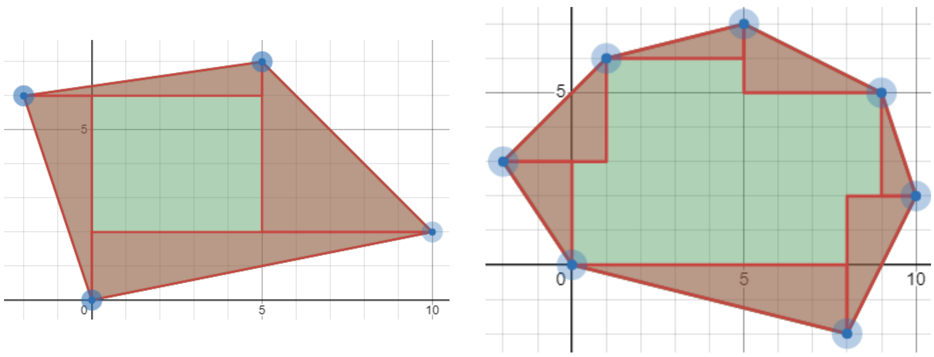


3 Implementation in Schools

The shoelace formula is not widely taught in schools. Instead, students are taught to find the area of polygons by computing the areas of triangles whose hypotenuses lie along their edges, summing them up, and adding their combined area to whatever remains.

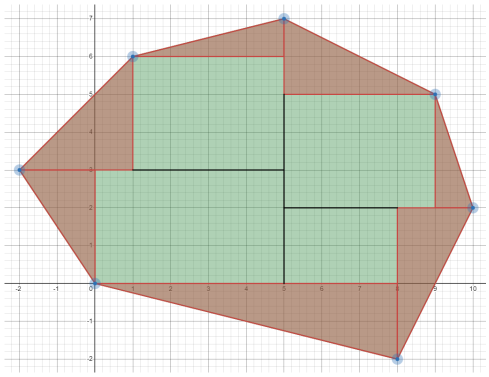
Students will quickly realize that for polygons with very few vertices, their ordinary method suffices, but when applied to polygons with more vertices, the method falls apart, becoming tedious and cumbersome to employ. This is suggested in Figure 3.

Figure 3: Manually computing the sum of the red area and the green area to find that of the whole shape becomes tedious the more vertices are involved.



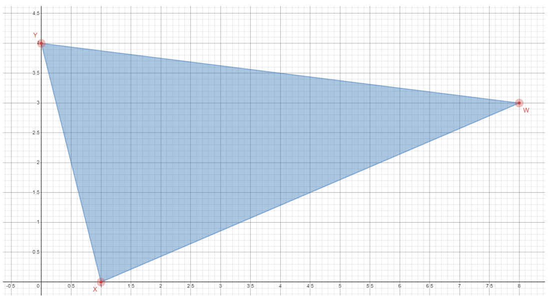
In addition to the increased number of triangles the ordinary method involves, the area in between them tends to become more detailed the higher the number of vertices the polygon consists of. This means students are not only obligated to calculate the areas of the triangles on the edges of the polygon, they must also dissect the remaining portion into rectangles to compute its area as shown in Figure 4.

Figure 4: The black lines split the green interior into rectangles whose areas can be found using formulae, unlike the area of the interior itself.



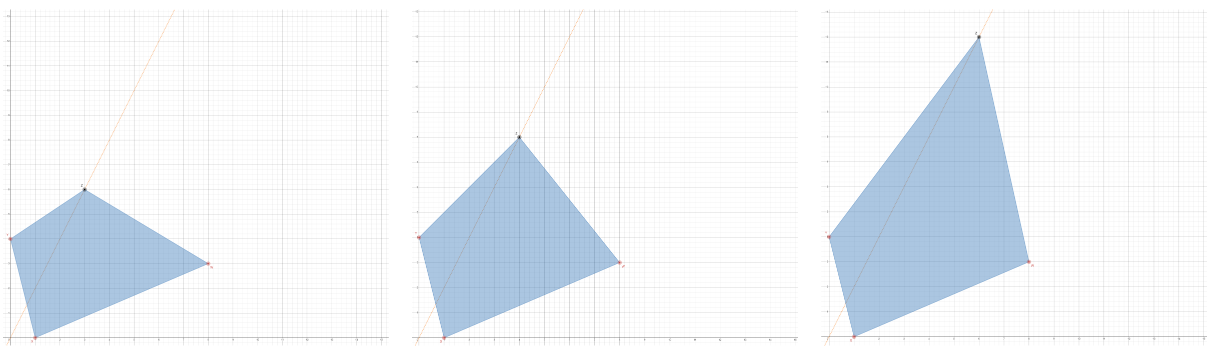
The shoelace formula also becomes especially versatile in cases involving variables. Consider the polygon in Figure 5, comprising points W, X, and Y defined at $(8,3)$, $(1,0)$, and $(0,4)$ respectively.

Figure 5: A starter polygon with three vertices.



When a fourth point Z is introduced that lies on the line $y = 2x$, a variable is necessary to express the area of quadrilateral $WXYZ$. Adding triangles together becomes even more unwieldy because the position of Z can change, as suggested in Figure 6.

Figure 6: Creating quadrilaterals by adding a fourth vertex from the line $y = 2x$.

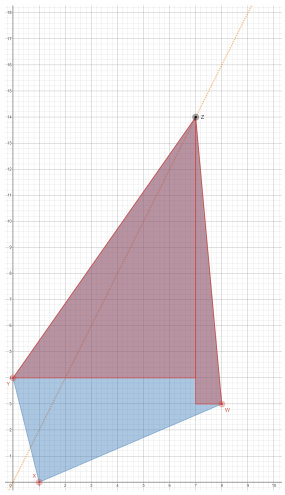


Z at $(3,6)$ | Z at $(4,8)$ | Z at $(6,12)$

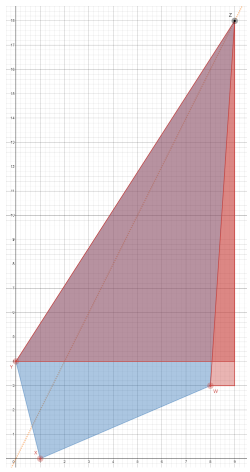
Worse still, when the point crosses the vertical or horizontal line defined by the x- or y-value of either point next to it, there exists no right triangle able to stay perfectly within the original shape as depicted in Figure 7.

By expressing the length of the horizontal leg of a triangle whose vertices include a variable point as the x- or y- value of any given minus the x- or y- value of its succeeding vertex, the leg length will be negative when a separate set of triangles would otherwise be necessary, thereby returning the signed area at the end when the absolute value is taken to ensure more area is added than subtracted. This means students need only choose an order for the vertices to follow.

Figure 7: In the figure on the right-hand side, no right triangle containing W and Z can exist fully within $WXYZ$. As such, the area of the red triangle is subtracted should it lie outside the quadrilateral as its base length technically equals -1.



Z at $(7, 14)$



Z at $(9, 18)$

Ultimately, the school-taught approach for finding the area of an irregular polygon is just as slow in a polygon with vertices whose positions vary as it is in a polygon with vertices that do not move. Students will quickly recognize the slowness of the ordinary method of finding and summing triangles and adding their areas to the area left behind as well as the speed of the shoelace formula, which only involves plugging values in.

To educate students about the resourcefulness of the shoelace formula, present them with polygons whose vertices have defined positions in the Cartesian plane and instruct them to compute the areas thereof. Begin with a triangle, and progress to shapes with seven or more vertices. In order to shed light on the tedium of the ordinary method, ask the pupils how long the polygonal area computations took the more vertices the polygons comprised. Then, educate them about the shoelace formula, teach them how to use it, and then repeat the activity. Students will realize how efficient the shoelace formula is compared to the method to which they are accustomed.

In order to reinforce students' knowledge of the shoelace formula, educators should present them with problems that involve its use in their solving processes. Educators may present area problems that involve solving for the variable position of a point given other points and the area of the polygon they compose, depending on the algebraic capabilities of their students.

The next section includes applications educators may use to teach their pupils or reinforce their understanding of the shoelace formula.

4 Applications

Simple Applications

Application 1. Find the area of pentagon $ABCDE$ with vertices $A(1,1)$, $B(2,5)$, $C(6,8)$, $D(8,4)$, and $E(3,0)$.

Application 2. A farmer drafts a plan to enclose an area to build a field by marking it with tacks and strings on a map to represent corners and the fences between them. How much area will the farmer enclose if he places posts at $A(-20,5)$, $B(-10,25)$, $C(4,22)$, $D(9,14)$, $E(16,-12)$, and $F(0,-17)$? Assume one coordinate unit on the map equals one meter in real life.

Application 3. The vertices of a convex quadrilateral $PQRS$ are as follows: $P(0,1)$, $Q(0,0)$, $R(1,0)$, and S, which lies along the curve $y = x^2$. Find a) the formula for the area of the shape given an x-value for S, and b) the position of S when the area of the shape equals 200 square units.

Application 4. Find the area of pentagon $ABCDE$ in terms of a if the positions of the points are defined in increasing alphabetical order as $(-a,0)$, $(0,2a)$, $(3a,0)$, $(2a,-a)$, and $(a,-2a)$.

Advanced Applications

Application 5 (Problem 2.90, AMC 10 preparation book, Sedrakyan). Convex quadrilateral $ABCD$, with an area of 18 and vertex coordinates $A(1,1)$, $B(2,7)$, $C(m,n)$, $D(6,3)$ is drawn on the xy-coordinate plane. What is $m+n$?

Application 6 (AMC 12B, problem 10, 2016). A quadrilateral has vertices $P(a,b)$, $Q(b,a)$, $R(-a,-b)$, and $S(-b,-a)$, where a and b are integers with $a > b > 0$. The area of $PQRS$ is 16. What is $a+b$?

Application 7 (AMC 12A, problem 17, 2020). The vertices of a quadrilateral lie on the graph of $y = \ln x$, and the x-coordinates of these vertices are consecutive positive integers. The area of the quadrilateral is $\ln \frac{91}{90}$. What is the x-coordinate of the leftmost vertex?

Application 8 (AIME II, problem 3, 2017). A triangle has vertices $A(0,0)$, $B(12,0)$, and $C(8,10)$. The probability that a randomly chosen point inside the triangle is closer to vertex B than to either vertex A or vertex C can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Application 9 (AIME I, problem 2, 2000). Let u and v be integers satisfying $0 < v < u$. Let $A = (u,v)$, let B be the reflection of A across the line $y = x$, let C be the reflection of B across the y-axis, let D be the reflection of C across the x-axis, and let E be the reflection of D across the y-axis. The area of pentagon $ABCDE$ is 451. Find $u+v$.

An Application in Calculus

Application 10. The positions of points A, B, C, D, and E over time are defined by the parametric curves $(\cos(t-\frac{\pi}{4})-1,4)$, $(\cos(t-\frac{\pi}{4}),3)$, $(2\sin(2t)+4,2)$, $(\cos(t-\frac{\pi}{4})+5,0)$, and $(-2\sin(2t),0)$ respectively at all values of t. Find a) the greatest area the points can enclose and b) the values of t for which this area occurs.

5 Solutions of Applications

Solutions for Simple Applications

Application 1. Find the area of pentagon $ABCDE$ with vertices $A(1,1)$, $B(2,5)$, $C(6,8)$, $D(8,4)$, and $E(3,0)$.

Solution. Gauss' area formula (shoelace formula) for five points is:

$$A = \frac{1}{2}\left|\left(x_1 \cdot y_2 + x_2 \cdot y_3 + x_3 \cdot y_4 + x_4 \cdot y_5 + x_5 \cdot y_1\right) - \left(y_1 \cdot x_2 + y_2 \cdot x_3 + y_3 \cdot x_4 + y_4 \cdot x_5 + y_5 \cdot x_1\right)\right|.$$

Following the format of the shoelace formula, we find:

$$x_1 = 1, \quad y_1 = 1,$$
$$x_2 = 2, \quad y_2 = 5,$$
$$x_3 = 6, \quad y_3 = 8,$$
$$x_4 = 8, \quad y_4 = 4,$$
$$x_5 = 3, \quad y_5 = 0.$$

We can plug these values into the formula to get that

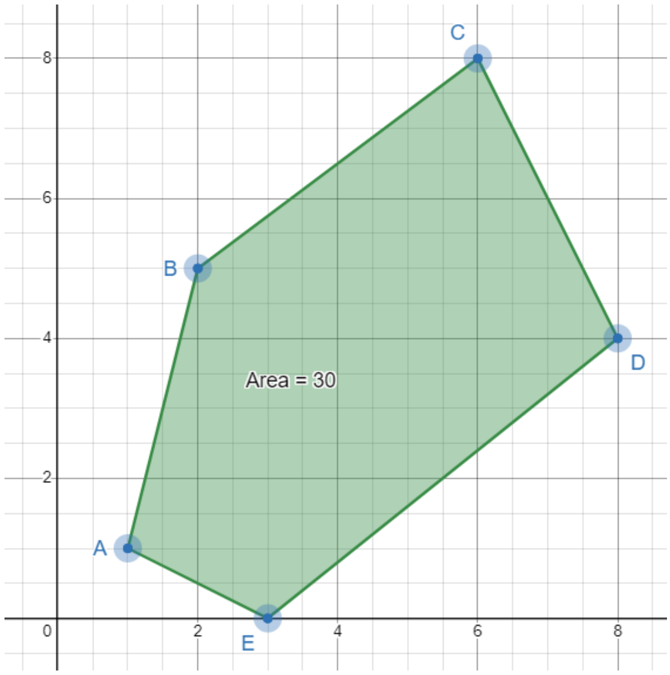
$$A = \frac{1}{2}\left|\left(1 \cdot 5 + 2 \cdot 8 + 6 \cdot 4 + 8 \cdot 0 + 3 \cdot 1\right) - \left(1 \cdot 2 + 5 \cdot 6 + 8 \cdot 8 + 4 \cdot 3 + 0 \cdot 1\right)\right|.$$

Simplifying, we get that

$$A = \frac{1}{2}|48 - 108|.$$

So, $A = 30$ square units, as illustrated in Figure 8.

Figure 8: $ABCDE$ in the Cartesian plane.



□

Application 2. A farmer drafts a plan to enclose an area to build a field by marking it with tacks and strings on a map to represent corners and the fences between them. How much area will the farmer enclose if he places posts at $A(-20,5)$, $B(-10,25)$, $C(4,22)$, $D(9,14)$, $E(16,-12)$, and $F(0,-17)$? Assume one coordinate unit on the map equals one meter in real life.

Solution. We can plug the values for each point to express the area as

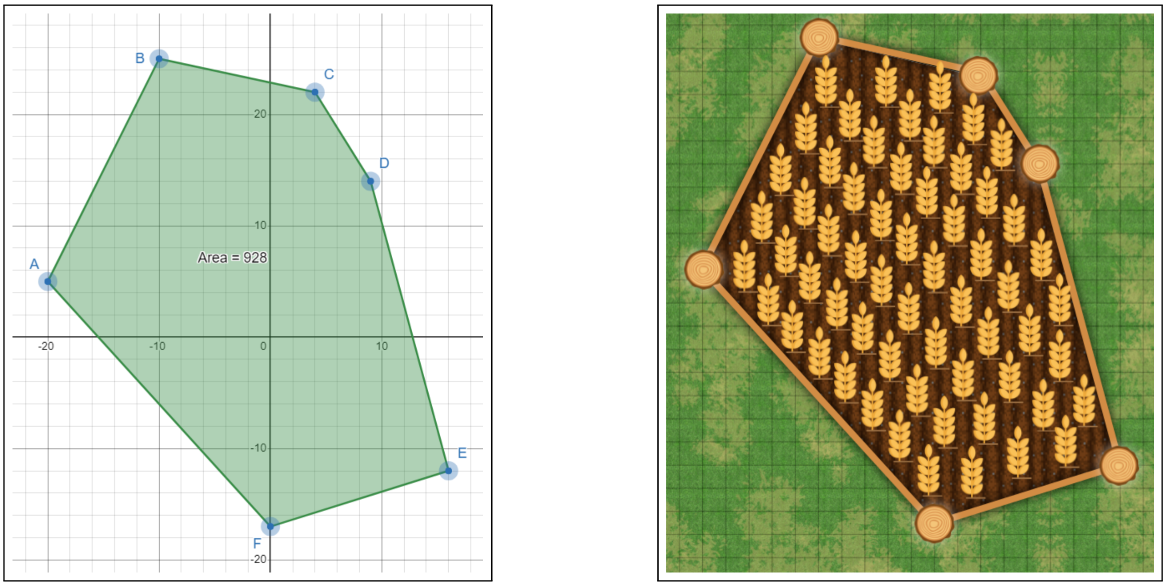
$$A = \frac{1}{2}\Big|\Big(-20\cdot 25 + (-10)\cdot 22 + 4\cdot 14 + 9\cdot(-12) + 16\cdot(-17) + 0\cdot 5\Big)$$
$$-\Big(5\cdot(-10) + 25\cdot 4 + 22\cdot 9 + 14\cdot 16 + (-12)\cdot 0 + (-17)\cdot(-20)\Big)\Big|$$

which simplifies to

$$A = \frac{|-1044 - 812|}{2}.$$

So, $A = 928$ square meters, as illustrated in Figure 9(a).

Figure 9: Polygon $ABCDE$.



(a) $ABCDE$ in the Cartesian plane. **(b)** $ABCDE$ on a map.

□

Application 3. The vertices of a convex quadrilateral $PQRS$ are as follows: $P(0,1)$, $Q,(0,0)$, $R(1,0)$, and S, which lies along the curve $y = x^2$. Find a) the formula for the area of the shape given an x-value for S, and b) the position of S when the area of the shape equals 200 square units.

Solution a. Because point S lies along $y = x^2$, we can express it as the ordered pair (t, t^2), and then express the area of $PQRS$ in terms of t. Plugging all the points into the shoelace formula, we obtain the following equation:

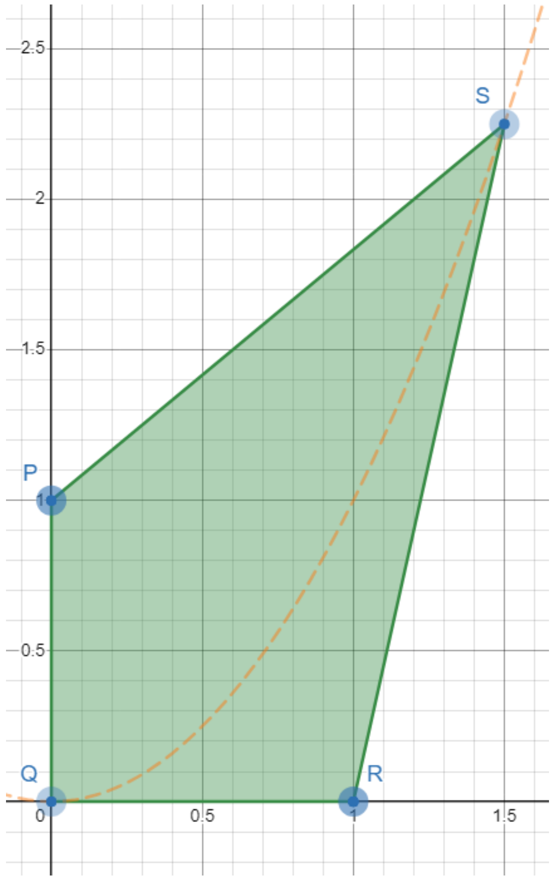
$$A = \frac{1}{2}\Big|\Big(0+0+t^2+t\Big) - \Big(0+0+0+0\Big)\Big|.$$

Simplifying, we get

$$A = \frac{1}{2}|t^2 + t|.$$

The case of $PQRS$ when $t = 1.5$ is illustrated in Figure 10.

Figure 10: $PQRS$ when $t = 1.5$.



Solution b. Using the equation we previously obtained, we can plug in 200 for the area and solve for t. Doing so gives us

$$200 = \frac{1}{2}|t^2 + t|.$$

The shape is only convex when t lies on or to the right of $\frac{-3+\sqrt{17}}{2}$, as $t = \frac{-3+\sqrt{17}}{2}$ when $Q(t, t^2)$ lies on $PR = -x + 1$. The zeros of the function within the absolute value lines, $x^2 + x$, are $x = 0$ and $x = -1$, but since the x-value of S is greater than or equal to $\frac{-3+\sqrt{17}}{2}$, which itself is greater than the rightmost zero, $x = 0$, we need not consider how the inner function x^2+x is affected when crossing its zeros and changing from positive to negative, or vice versa.

Because of this, we can effectively replace the absolute value lines with parentheses, which gives us

$$200 = \frac{1}{2}(t^2 + t).$$

Simplifying the expression allows us to find that

$$200 = \frac{1}{2}t^2 + \frac{1}{2}t.$$

Multiplying both sides by 2 and moving everything to the right-hand side of the equals sign, we get

$$0 = t^2 + t - 400$$

and solving with the quadratic formula tells us that

$$t = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-400)}}{2}.$$

Recall that $\frac{-1-\sqrt{1601}}{2}$ is negative, so the only possible value for t is $\frac{-1+\sqrt{1601}}{2}$.

Substituting our newfound t-value into the ordered pair $S(t,t^2)$ tells us that S lies at

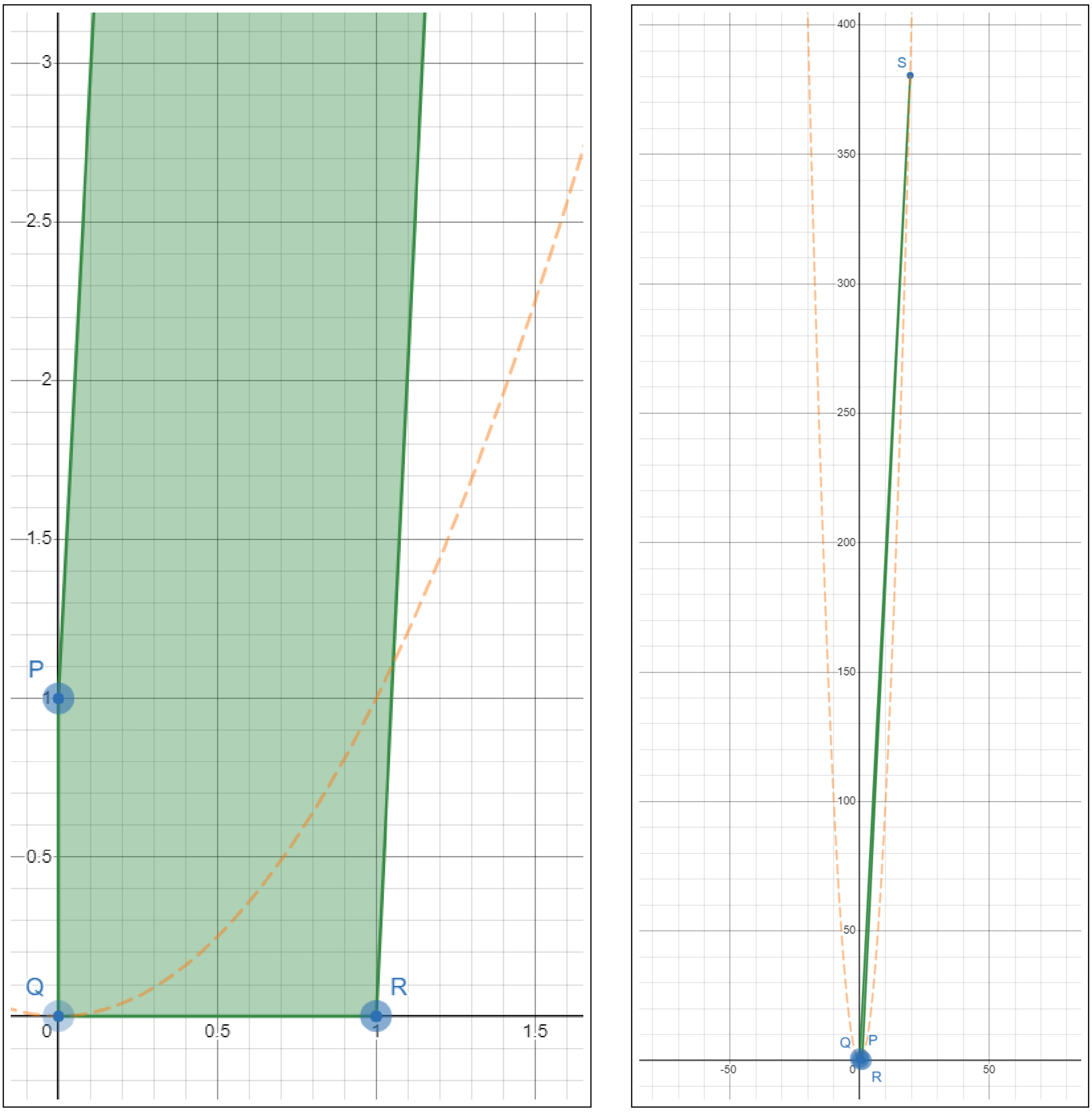
$$\Big(\frac{-1+\sqrt{1601}}{2}, \frac{1-2\sqrt{1601}+1601}{4}\Big).$$

Simplifying gives us our solution, or the following ordered pair:

$$S\Big(-\frac{1}{2}+\frac{\sqrt{1601}}{2}, \frac{801}{2}-\frac{\sqrt{1601}}{2}\Big).$$

Two plots of $PQRS$ are illustrated in Figure 11.

Figure 11: Two views of $PQRS$.



(a) $PQRS$ from a relatively small scale. **(b)** $PQRS$ entirely in one image.

□

Application 4. Find the area of pentagon $ABCDE$ in terms of a if the positions of the points are defined in increasing alphabetical order as $(-a, 0)$, $(0, 2a)$, $(3a, 0)$, $(2a, -a)$, and $(a, -2a)$.

Solution. We can immediately plug points A, B, C, D, and E into the shoelace formula to find that

$$A = \frac{1}{2}\Big|\Big(- 2a^2 + 0 - 3a^2 - 4a^2 + 0\Big) - \Big(0 + 6a^2 + 0 - a^2 + 2a^2\Big)\Big|.$$

Simplifying the expression, we get that

$$A = \frac{|-16a^2|}{2}.$$

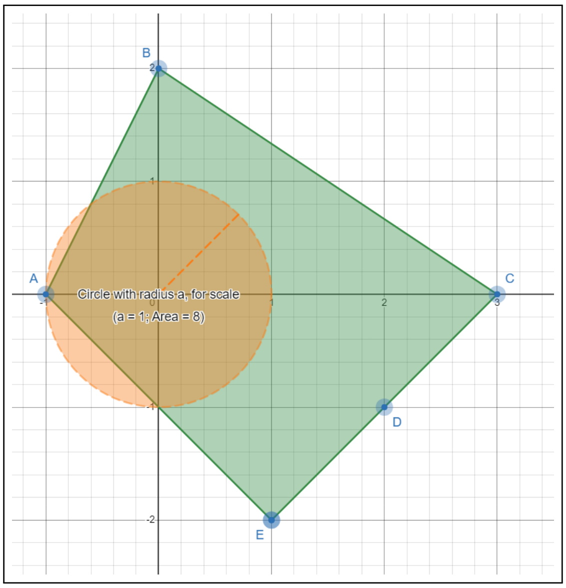
Due to the function within the absolute value bars, $f(a) = -16a^2$, having a zero at the origin, we need to analyze the signs of the function in order to eliminate the absolute value symbols from the equation, as doing so will make the expression easier to work with and there will be no casework needed afterward. Plugging an arbitrary value to the left of $a = 0$ into the inner function (in this case, we will select $a = -1$, which returns $f(a) = -16$) lets us know that the function is negative along the interval $(-\infty, 0)$. Doing the same for an a-value to the right of $a = 0$ (in this case, we will select $a = 1$, which returns $f(a) = -16$) lets us know that the function is negative along the interval $(0, \infty)$.

Because the absolute value of any negative number u is the positive number $-u$, we can find that $\frac{1}{2}|-16a^2| = \frac{1}{2} \cdot -(-16a^2)$ along the intervals $(-\infty, 0)$ and $(0, \infty)$ because the function is negative along those intervals. We can also declare such a statement to be true along the interval $(0, 0)$ because the interval is simply a zero of the inner function, and $-(0) = 0$. As such, we declare the following to be true for all values of a:

$$A = -\frac{1}{2}(-16a^2)$$

Simplifying further leaves us with our final expression, $A = 8a^2$.

Figure 12: $ABCDE$ in the Cartesian plane. The orange circle, which exists purely for scale, is centered at the origin and has radius a. In this example, $a = 1$ and $A = 8$.



□

Solutions for Advanced Applications

Application 5 (Problem 2.90, AMC 10 preparation book, Sedrakyan). Convex quadrilateral $ABCD$, with an area of 18 and vertex coordinates $A(1,1)$, $B(2,7)$, $C(m,n)$, $D(6,3)$, is drawn on the $xy-$coordinate plane. What is $m+n$?

Solution. We have the points $A(1,1)$, $B(2,7)$, $C(m,n)$, and $D(6,3)$; by means of shoelace formula, we get

$$\frac{1}{2}\Big|\Big(7\cdot 1+3\cdot m+2\cdot n+6\cdot 1\Big)-\Big(2\cdot 1+7\cdot m+6\cdot n+3\cdot 1\Big)\Big|=18.$$

Simplifying, we get

$$|-4m-4n+8|=36.$$

This means that either

$$-4m-4n+8=36,$$

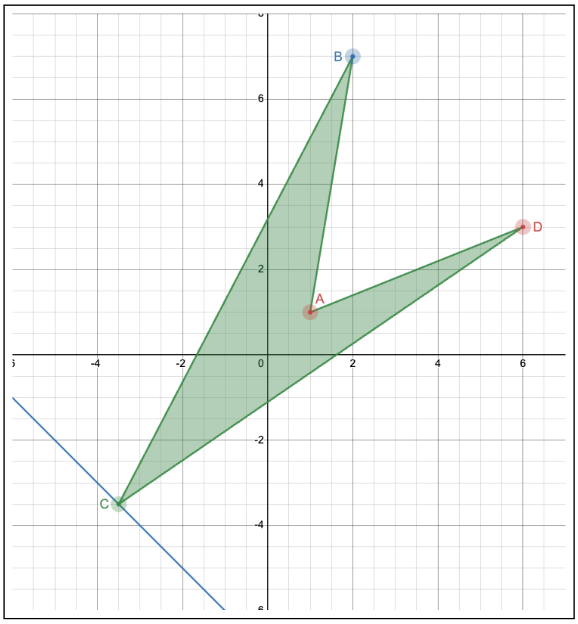
or

$$-4m-4n+8=-36.$$

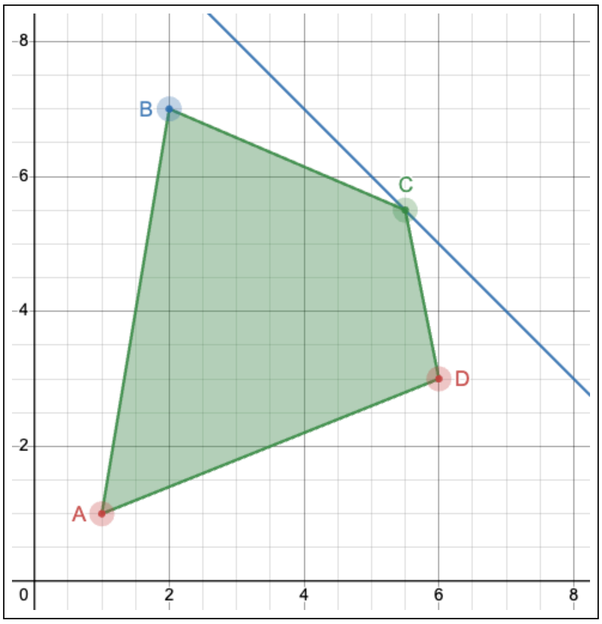
Thus, it follows that either $m+n=-7$ or $m+n=11$.

The case $m+n=-7$ is extrinsic, because if point C lies on the line $x+y=-7$, then none of the points on $x+y=-7$ make $ABCD$ convex. This is illustrated in Figure 13(a). Similarly, the solution $m+n=11$ is correct, because if point C lies on the line $x+y=11$, then some points on $x+y=11$ make $ABCD$ convex, as illustrated in Figure 13b.

Figure 13: Illustration of extrinsic and intrinsic cases for polygon $ABCD$.



(a) The case $m+n=-7$ is extrinsic.



(b) A case in which $ABCD$ is convex.

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Application 6 (AMC 12B, problem 10, 2016). A quadrilateral has vertices $P(a,b)$, $Q(b,a)$, $R(-a,-b)$, and $S(-b,-a)$, where a and b are integers with $a > b > 0$. The area of $PQRS$ is 16. What is $a+b$?

Solution. As the area is 16, then Gauss' area formula (the shoelace formula) can be written as follows:

$$\frac{1}{2}\Big|\Big(a^2-b^2+a^2-b^2\Big)-\Big(b^2-a^2+b^2-a^2\Big)\Big|=16.$$

Simplifying, we get

$$|4a^2-4b^2|=32.$$

This means that either

$$a^2-b^2=8, \text{ or } a^2-b^2=-8.$$

As $a > b > 0$, then $a^2-b^2>0$. So $a^2-b^2=8$. We get

$$(a+b)\cdot(a-b)=8.$$

So, either

$$\begin{cases} a+b=8, \\ a-b=1, \end{cases}$$

or

$$\begin{cases} a+b=4, \\ a-b=2, \end{cases}$$

As a and b are positive integers, then we get $a=3$ and $b=1$.
$a+b=1+3=4$. □

Application 7 (AMC 12A, problem 17, 2020). The vertices of a convex quadrilateral lie on the graph of $y=\ln x$, and the x-coordinates of these vertices are consecutive positive integers. The area of the quadrilateral is $\ln\frac{91}{90}$. What is the $x-$coordinate of the leftmost vertex?

Solution. We can denote the x-coordinate of the leftmost vertex as x. Using the shoelace formula, we get that

$$\frac{1}{2}\Big|\Big(x\ln(x+1)+(x+1)\ln(x+2)+(x+2)\ln(x+3)+(x+3)\ln(x)\Big)-$$

$$-\Big((x+1)\ln(x)+(x+2)\ln(x+1)+(x+3)\ln(x+2)+x\ln(x+3)\Big)\Big|=\ln\frac{91}{90}.$$

We get

$$\frac{1}{2}\Big|x\ln(x+1)+(x+1)\ln(x+2)+(x+2)\ln(x+3)+(x+3)\ln(x)-$$

$$-(x+1)\ln(x)-(x+2)\ln(x+1)-(x+3)\ln(x+2)-x\ln(x+3)\Big|=\ln\frac{91}{90}.$$

So, we have

$$\frac{1}{2}\Big|-2\ln(x+1)-2\ln(x+2)+2\ln(x+3)+2\ln(x)\Big|=\ln\frac{91}{90}.$$

This means that, either

$$\frac{1}{2}\cdot 2\cdot\ln\Big(\frac{x^2+3x}{x^2+3x+2}\Big)=\ln\frac{91}{90},$$

or

$$\frac{1}{2}\cdot 2\cdot\ln\Big(\frac{x^2+3x}{x^2+3x+2}\Big)=-\ln\frac{91}{90}.$$

As such, either

$$0 = x^2 + 3x + 182$$

or

$$0 = x^2 + 3x - 180,$$

meaning that either

$$x = -\frac{3}{2} + i\frac{\sqrt{719}}{2}, \quad x = -\frac{3}{2} - i\frac{\sqrt{719}}{2},$$

or

$$x = 12, \quad x = -15.$$

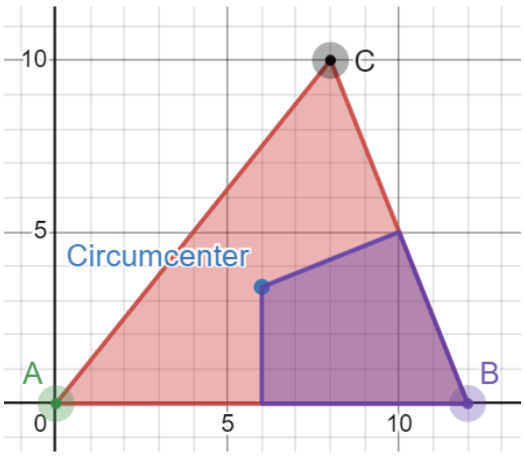
The solution to the problem is $x = 12$, the only positive integer solution among the four possible answers. □

Application 8 (AIME II, problem 3, 2017). A triangle has vertices $A(0,0)$, $B(12,0)$, and $C(8,10)$. The probability that a randomly chosen point inside the triangle is closer to vertex B than to either vertex A or vertex C can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.

Solution. Assume a random point P. For any pair of points X and Y, draw a line segment connecting them and a perpendicular bisector intersecting the line segment. P will be closer to X if it lies on the same side of the perpendicular bisector as X. The same is true for Y.

We can use this postulate to draw the shape that includes only the points inside $\triangle ABC$ that are closest to point B. Draw the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$, and denote them as segments a and b respectively. All points inside $\triangle ABC$ that are on the same side of both a and b as point B are closer to B than to any other vertex of $\triangle ABC$. The resulting shape is a quadrilateral Q, depicted as a purple region in Figure 14.

Figure 14: Points in $\triangle ABC$ closer to vertex B than any other vertex are in the purple region.



Next, we find the area of Q. The vertices of Q are $(6, 3.4)$, $(10, 5)$, $B(12, 0)$, and $(6, 0)$. Using Gauss's area formula (the shoelace formula), we get

$$A = \frac{1}{2}\left|\left(6\cdot 5 + 10\cdot 0 + 12\cdot 0 + 6\cdot 3.4\right) - \left(3.4\cdot 10 + 5\cdot 12 + 0\cdot 6 + 0\cdot 6\right)\right|.$$

Simplifying, we get $A = \frac{1}{2}|-43.6|$. This means that $A = 21.8$. The area of $\triangle ABC$ is $\frac{1}{2}\cdot 12\cdot 10 = 60$. To get our answer, we rewrite $\frac{21.8}{60}$ as $\frac{109}{300}$. We have $109+300 = 409$. So, the answer is 409. □

Application 9 (AIME I, problem 2, 2000). Let u and v be integers satisfying $0 < v < u$. Let $A = (u, v)$, let B be the reflection of A across the line $y = x$, let C be the reflection of B across the y-axis, let D be the reflection of C across the x-axis, and let E be the reflection of D across the y-axis. The area of pentagon $ABCDE$ is 451. Find $u + v$.

Solution. The five points are located in the following positions: $A(u, v)$, $B(v, u)$, $C(-v, u)$, $D(-v, -u)$, and $E(v, -u)$. Using Gauss's area formula (the shoelace formula), we get

$$451 = \frac{1}{2}\left|\left(u^2 + uv + uv + uv + v^2\right) - \left(u^2 - uv - uv - uv - u^2\right)\right|.$$

Simplifying, we get

$$902 = |2u^2 + 6uv|.$$

This means either

$$902 = 2u^2 + 6uv$$

or

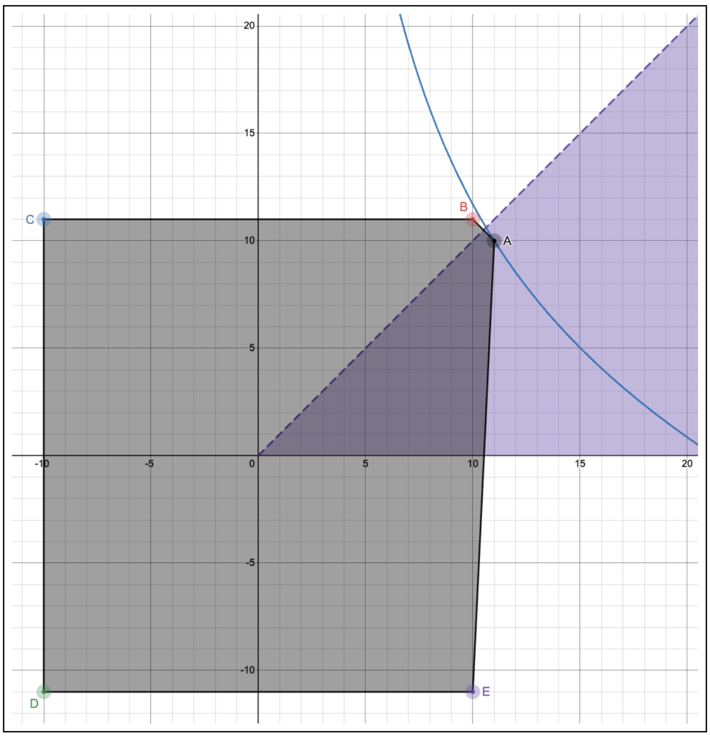
$$-902 = 2u^2 + 6uv.$$

However, only the former is true and the latter is false (as $0 < v < u$, we get that neither u nor v are negative, meaning $2u^2 + 6uv$ cannot be negative; thus, the latter is false). Solving for v, we get

$$v = \frac{451 - u^2}{3u}.$$

where $u > 0$. This gives us the curve on which A must lie in order for the area of $ABCDE$ to equal 451. Denote this curve as $V(u)$ as shown in Figure 15.

Figure 15: The blue curve represents $V(u)$; the purple area represents the region where point A can lie.



Because $0 < v < u$, point A must lie below the line $v = u$ but above the line $v = 0$. Denote the area where point A can be as X. As such, we only have to search the interval of $V(u)$ that lies in X. The bounds of this interval can be determined by finding where $V(u)$ intersects $v = u$ and where it intersects $v = 0$. We get that

$$u = \frac{451 - u^2}{3u}$$

and that

$$0 = \frac{451 - u^2}{3u}.$$

Simplifying for each respective equation, we get

$$u = \pm\frac{\sqrt{451}}{2}; u = \pm\sqrt{451}.$$

Because $0 < v < u$, we get the bounds of our interval: $(\frac{\sqrt{451}}{2}, \sqrt{451})$, or approximately $(10.618, 21.237)$. Now, all that is left to do is to find an integer value of u for which $V(u)$ is also an integer.

We get that $u = 11$ and that $v = 10$. This leads us to our answer: 11+10=21. □

Solutions for Application in Calculus

Application 10. The positions of points A, B, C, D, and E over time are defined by the parametric curves $(\cos(t - \frac{\pi}{4}) - 1, 4)$, $(\cos(t - \frac{\pi}{4}), 3)$, $(2\sin(2t) + 4, 2)$, $(\cos(t - \frac{\pi}{4}) + 5, 0)$, and $(-2\sin(2t), 0)$ respectively at all values of t. Find a) the greatest area the points can enclose and b) the values of t for which this area occurs.

Solution a. Plugging the x- and y-values of points A, B, C, D, and E into the shoelace formula, we obtain the following expression:

$$A = \frac{1}{2}\Big|\Big(\Big(3\cos\Big(t - \frac{\pi}{4}\Big) - 3\Big) + \Big(2\cos\Big(t - \frac{\pi}{4}\Big)\Big) + 0 + 0 + \Big(-8\sin(2t)\Big)\Big)$$
$$-\Big(\Big(4\cos\Big(t - \frac{\pi}{4}\Big)\Big) + \Big(6\sin(2t) + 12\Big) + \Big(2\cos\Big(t - \frac{\pi}{4}\Big) + 10\Big) + 0 + 0\Big)\Big|.$$

Simplifying this expression, we get that

$$A = \frac{1}{2}\Big|\Big(5\cos\Big(t - \frac{\pi}{4}\Big) - 3 - 8\sin(2t)\Big) - \Big(6\cos\Big(t - \frac{\pi}{4}\Big) + 6\sin(2t) + 22\Big)\Big|$$

and that

$$A = \frac{1}{2}\Big| - \cos\Big(t - \frac{\pi}{4}\Big) - 25 - 14\sin(2t)\Big|.$$

To remove the absolute value symbols, we must conduct a sign analysis of the function within them. We denote the inner function as $f(t) = -\cos(t - \frac{\pi}{4}) - 25 - 14\sin(2t)$, and will next prove that $f(t)$ has no zeros. We can do so by attempting to solve for $f(t) = 0$, which gives us

$$0 = \cos\Big(t - \frac{\pi}{4}\Big) - 25 - 14\sin(2t).$$

Adding 25 to both sides allows us to find that

$$25 = \cos\Big(t - \frac{\pi}{4}\Big) - 14\sin(2t).$$

The values of sine and cosine never exceed 1, meaning that even if $\sin(2t)$ equated to -1 and $\cos(t-\frac{\pi}{4})$ equated to 1 at some value of t, the value of the expression to the right of the equals sign would only ever reach $14+1=15$. In other words, $\cos(t-\frac{\pi}{4})-14\sin(2t) \leq 15 < 25$. This implies that

$$\cos\left(t-\frac{\pi}{4}\right)-14\sin(2t) < 25$$

and that

$$\cos\left(t-\frac{\pi}{4}\right)-25-14\sin(2t) < 0.$$

Hence, $f(t) < 0$ and will never equal 0, meaning $f(t)$ has no zeros along the entire function. Because of this, we can plug in an arbitrary value to determine whether $f(t)$ is positive or negative along the interval $(-\infty, \infty)$. For simplicity's sake, we will select $t = 0$, which returns $f(t) = -\frac{\sqrt{2}}{2} - 25 \approx -25.707$. Thus, $f(t)$ is negative along the entire function, meaning its absolute value is itself multiplied by -1 as the absolute value of any negative number u equals $-u$. In removing the absolute value symbols, we obtain:

$$A = \frac{1}{2}(-1)(-\cos\left(t-\frac{\pi}{4}\right)-25-14\sin(2t))$$

which simplifies to

$$A = \frac{1}{2}\cos\left(t-\frac{\pi}{4}\right)+\frac{25}{2}+7\sin(2t).$$

We have now found an expression for the area of $ABCDE$ that is easy to work with as it no longer involves absolute value. To find the maximum area of the shape, we must find the zeros of the derivative of our newfound expression for its area with respect to t. We first find the derivative of the expression itself:

$$\frac{dA}{dt} = \frac{1}{2}\Big(-\sin\left(t-\frac{\pi}{4}\right)\cdot 1\Big)+0+7(\cos(2t)\cdot 2),$$

which simplifies to

$$\frac{dA}{dt} = -\frac{1}{2}\sin\left(t-\frac{\pi}{4}\right)+14\cos(2t).$$

Plugging in 0 for $\frac{dA}{dt}$ to find the maxima and minima, we get that

$$0 = -\frac{1}{2}\sin\left(t-\frac{\pi}{4}\right)+14\cos(2t)$$

and that

$$\frac{1}{2}\sin\left(t-\frac{\pi}{4}\right) = 14\cos(2t).$$

By employing the cofunction identity of sine (i.e., $\cos(\theta) = \sin(\frac{\pi}{2}-\theta)$), we can substitute $\sin(\frac{\pi}{2}-2t)$ for $\cos(2t)$, which gives us

$$\frac{1}{2}\sin\left(t-\frac{\pi}{4}\right) = 14\sin\left(\frac{\pi}{2}-2t\right).$$

Using the negative identity of sine (i.e., $\sin(-\theta) = -\sin(\theta)$) and factoring 2 from the inside of the parentheses of the expression on the right, we get that

$$-\frac{1}{2}\sin\left(\frac{\pi}{4}-t\right) = 14\sin(2\Big(\frac{\pi}{4}-t\Big)).$$

We can let u equal $\frac{\pi}{4}-t$ to simplify the equation, which gives us

$$-\frac{1}{2}\sin(u) = 14\sin(2u).$$

The double angle identity for sine (i.e., $\sin(\theta) = 2\sin(\theta)\cos(\theta)$) allows us to find that

$$-\frac{1}{2}\sin(u) = 28\sin(u)\cos(u).$$

Moving everything to one side and multiplying by 2, we get that

$$0 = 56\sin(u)\cos(u) + \sin(u),$$

and factoring gives us the following:

$$0 = \sin(u)(56\cos(u) + 1).$$

This means that either $\sin(u) = 0$, $56\cos(u) - 1 = 0$, or both. We can substitute $\frac{\pi}{4} - t$ back in for u and use simple algebra to find:

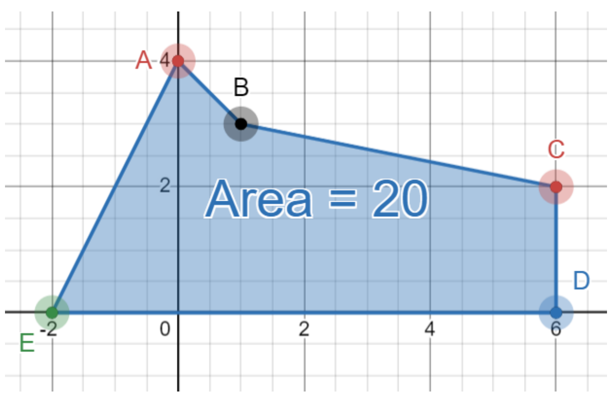
$$\begin{cases} t = \frac{\pi}{4} + \pi n, n \in \mathbb{Z} \\ t = \frac{\pi}{4} + \arccos(-\frac{1}{56}) + 2\pi n, n \in \mathbb{Z} \\ t = \frac{\pi}{4} - \arccos(-\frac{1}{56}) + 2\pi n, n \in \mathbb{Z} \end{cases}$$

Because the derivative in all of these cases is zero, these yield either maxima or minima of the area equation, meaning these are the only points we must find the corresponding area for. Since the area function oscillates, at least one of the t-values we have found corresponds to a maximum of the function for area. To find the maximum area given t from these three cases, we can plug them into our area formula, which we will write as a function, $A(t) = \frac{1}{2}\cos(t - \frac{\pi}{4}) + \frac{25}{2} + 7\sin(2t)$. We will plug in 0 for n in each case as well as 1 for n in the first case to make sure all of the cases that arise within the period of the function (2π) are examined. Doing so gives us the following:

$$\begin{cases} A(\frac{\pi}{4}) = 20 \\ A(\frac{5\pi}{4}) = 19 \\ A\Big(\frac{\pi}{4} + \arccos(-\frac{1}{56})\Big) = A\Big(\frac{\pi}{4} - \arccos(-\frac{1}{56})\Big) \approx 5.496 \end{cases}$$

Solution b. $A = 20$ square units corresponds to $t = \frac{\pi}{4} + 2\pi n, n \in \mathbb{Z}$. Figure 16 illustrates the shape with the t-value that maximizes area.

Figure 16: $ABCDE$ in the Cartesian plane when $t = \frac{\pi}{4}$.



□

6 Closing Statement

As the above contents have demonstrated, Gauss's area formula for irregular shapes is not only versatile, but is also easy to teach to students, even easier to apply to problems, and much simpler and more efficient at computing the area of an irregular shape than the method many students are familiar with. More importantly, it is also applicable to a myriad of levels of mathematical expertise spanning from middle school algebra to college-level calculus. For these reasons, Gauss' area formula for irregular shapes ought to receive a place in the school curriculum alongside other formulae taught to young students learning about the coordinate plane.

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