Unpacking the Ambiguous Case to Develop Conceptual Knowledge and Representational Competence

***Kyle T. Schultz**, University of Mary Washington*

Abstract: *The ambiguous case is a trigonometry topic for which high school students are often told "stay away from angle-side-side." In many cases, however, these students do not get the opportunity to explore the underlying mathematical context that serves as the basis of this warning. After a briefly presenting an overview of the ambiguous case, the author describes a mathematical activity using simple homemade manipulatives to support secondary and post-secondary students' work to unpack and understand the ambiguous case. It then discusses how this activity can support teachers' understanding of representational competence and recommends general practices supporting students' purposeful and effective use of mathematical representations.*

Keywords: *Ambiguous case, representation, pre-service teachers, trigonometry*

Introduction

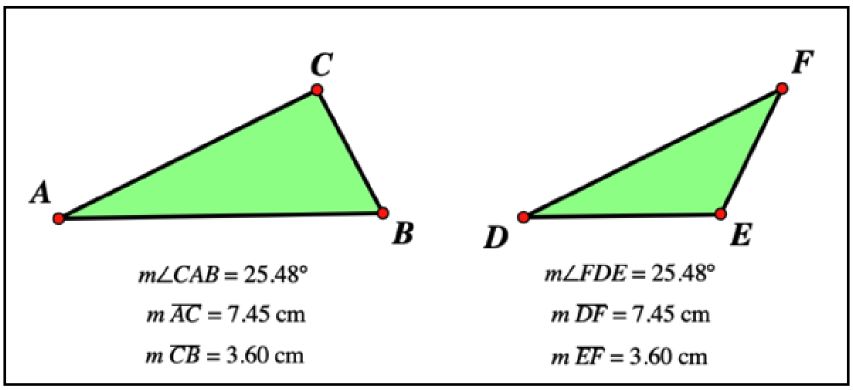
As mathematics teachers, we want our students to become proficient in ways that will enable them to think flexibly about mathematics content so they may apply it in novel situations. The National Council of Teachers of Mathematics (NCTM, 2000) process standards and the Common Core *Standards for Mathematical Practice* (CCSSO, 2010) provided frameworks for what this kind of proficiency would look like and how it might develop. In particular, one of the process standards, Representation, plays a key role in how people understand mathematical ideas. Although many think of representations (such as equations, tables, graphs, and diagrams) as the tangible products containing these ideas, the skills of expressing a mathematical idea using these forms and using them advantageously to solve problems is equally important (NCTM, 2014). Novick (2004) parsed out these skills and described them as *representational competence*.

This article examines prospective secondary mathematics teachers' introduction to the construct of representational competence through hands-on exploration of the ambiguous case, a trigonometry topic for which high school students are often told "stay away from angle-side-side," but do not get the opportunity to explore or develop an understanding of the underlying mathematical context that serves as the basis of this warning. In this article, I present a brief overview of the ambiguous case, describe a mathematical activity using simple homemade manipulatives and supporting secondary and post-secondary students' work to unpack and understand the ambiguous case, and conclude with a rationale for how this activity can support prospective teachers' understanding of representational competence.

The Ambiguous Case

In geometry, the ambiguous case occurs when the known corresponding measurements, when comparing two triangles, consist of two sides and an angle opposite to one of those sides (the "non-included angle"), shortened to "side-side-angle" or SSA. Some SSA configurations do not result in one specific triangle (Figure 1). Typically, secondary students first become acquainted with the ambiguous case in an introductory geometry course when studying methods to prove triangle congruence. As they learn combinations of corresponding side and angle measurements that can demonstrate congruence (e.g., angle-angle-side congruence postulate), students are also warned not to use SSA because this configuration sometimes produces a false positive situation in which the two triangles in question are demonstrated to be congruent when they actually are not. For the purpose of mastering the skill of proving triangles congruent, teachers frequently deem this warning and explanation as sufficient and conclusive.[1]

Figure 1: *Two distinct triangles constructed using the same given SSA measurements.*



Later, in the context of trigonometry, students develop the skill of solving triangles (determining the measures of all sides and angles of a triangle, given some of these measures) often by using tools such as the law of sines and law of cosines. In this context, the ambiguous case again becomes significant because, in SSA situations, students cannot depend on the existence of a single solution, which only occurs 50% of the time for a randomly selected set of given SSA measurements (Yeshurun & Kay, 1983). Instead, they must analyze the relationships between the given measures to determine whether zero, one, or two solutions exist.

To prepare students to solve for any number of solutions, common practice often involves presenting a series of specific cases related to the type of angle and relative lengths of the sides students can use to guide their decision making about the number of solutions. Case (1989) presented a flowchart through which students can operationalize these cases. Other methods involve students using the law of sines to calculate an angle measure (if it exists) and analyze it in the context of the given angle measure to determine the number of solutions. These methods rely heavily, and often solely, on symbolic manipulation and rote memorization (Levine, 1987; Peek, 1987; West, 1992; Harrison, 2002; DeComo, 2008). Harrison (2002) recognized the need for some exploration of the geometric context by using "compasses, protractors, and straightedges to graph different combinations of two sides and a non-included angle on the Cartesian Plane" (p. 114) prior to performing the needed calculations.

A Conceptually-Focused Approach

My interest in exploring the ambiguous case with prospective secondary teachers came from my students' brainstorms of secondary mathematics topics they struggle to understand. Because this topic

[1] For an alternative and more conceptually-focused approach to introducing the ambiguous case in the context of triangle congruence, Cirillo et al. (2015) present an activity, *The Hidden Triangle Exploration*.

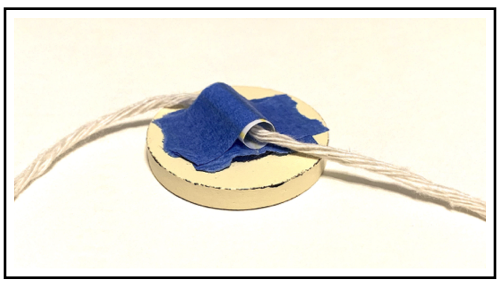
is less prominent in the literature and post-secondary curricula and yet requires integrating knowledge of a variety of fundamental concepts and skills in the secondary mathematics curriculum, it provides fertile ground for a robust activity modelling best practices in reasoning and sense-making as well as developing specialized content knowledge. To provide my students with an experience to develop deeper and more dynamic understanding of the ambiguous case, I created a hands-on collaborative activity in which they visually explored possible SSA scenarios. The prompt for the ambiguous case activity is as follows.

> In situations where two side lengths of a triangle and the measure of a non-included angle (SSA) are known: (1) How many solutions might the triangle have? (2) Under what conditions would that number of solutions be present? Organize and present your findings.

These students were secondary mathematics education candidates seeking licensure in grades 6–12. As part of their program, they enrolled in a mathematics methods course during their senior year, prior to their internship semester. Prior to this course, they had taken mathematics courses associated with their major and general education coursework with associated field experiences.

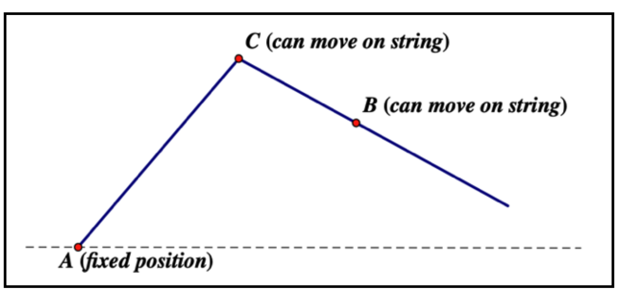
When facilitating this task, I provide each group of students white board space and a manipulative tool designed for exploration of the ambiguous case. This tool consists of a long piece of string attached to three circular magnets labeled A, B, and C. Magnet A is securely attached in a fixed position to one end of the string. Magnets C and B are attached in a manner that enables them to slide across the string (such as through a small piece of plastic straw taped to the magnet, as shown in Figure 2), enabling adjustments of position relative to magnet A and each other.

Figure 2: *An adjustable point (either B or C) represented with the manipulative tool, consisting of a magnet with a small segment of a straw taped to it. With this design, students can adjust the locations of B and C on the string as needed to experiment with different segment lengths.*



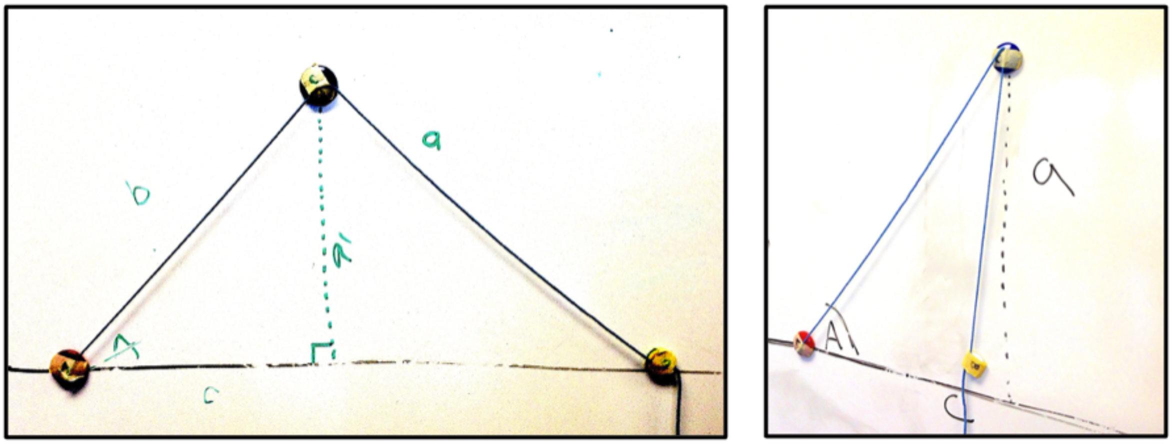
After drawing a horizontal line across the bottom of the white board to represent the path for one side of the triangle, students place magnet A on that line (Figure 3).

Figure 3: *The design of the manipulative tool consisting of three magnets serving as points A, B, and C on a string representing sides $\overline{AC}$ and $\overline{BC}$. Point A rests in a fixed position on the given horizontal line representing one side of the triangle, while students can adjust the locations of B and C.*



From this initial configuration, students then adjust the positions of magnets C and B in a variety of ways to model different measures for $\angle A$, side $\overline{AC}$, and side $\overline{BC}$ (Figure 4).

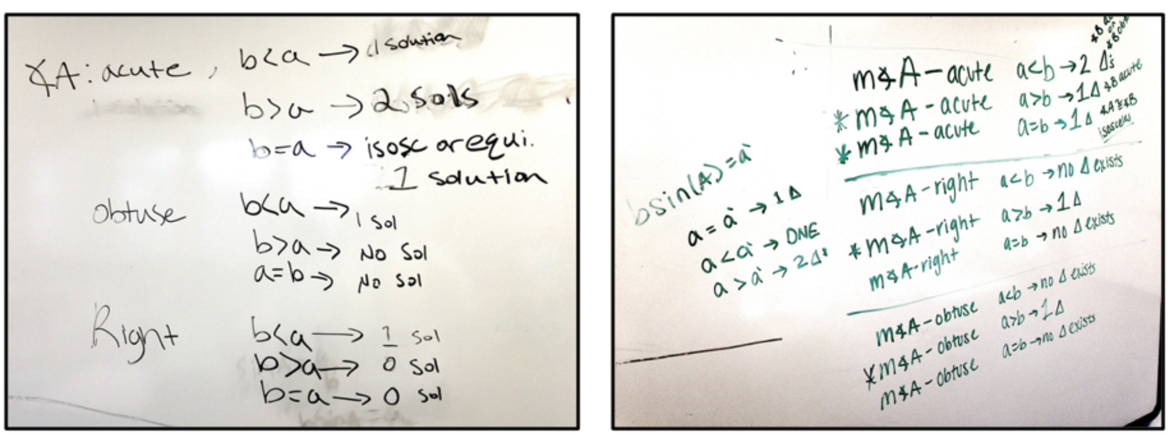
Figure 4: *(Left) The manipulative tool, placed to model a single-solution scenario with acute $\angle A$ and $a > b$; (Right) a no-solution scenario with acute $\angle A$ and $a < b \cdot sin(A)$.*



The structure of the activity prompts students to determine the important factors related to the segment and angle measures, create categories for each factor, ensure their categorization scheme included all possible SSA possibilities, and test each category for the number of solutions.

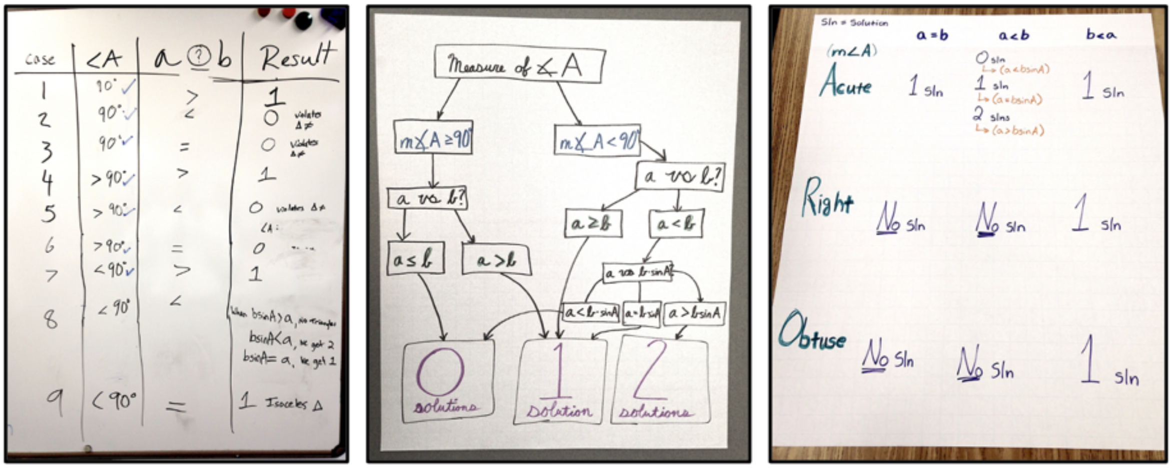
During my most recent implementation of this activity, most groups began by looking at different sizes for $\angle A$ (acute, right, obtuse) and then focused on the length of the sides. Figure 5 depicts two groups' efforts during this process, creating organized lists of cases grouped by angle type. In this process, some students noticed the parallel structure between the right- and obtuse-angle cases. As well, for the acute angle case, they needed to determine a way to formulate the length of the altitude of the triangle, $b \cdot sin(A)$ to facilitate comparison with values of BC.

Figure 5: *Two groups' work to define and analyze different cases of possible SSA measurements. The left-hand image depicts thinking that has yet to consider using the altitude length for comparison.*



My students considered different ways to organize and present their findings. Three different representations of the different ambiguous case scenarios were observed: an organized list, a flowchart (similar to that presented by Case, 1989), and a table (Figure 6).

Figure 6: *Three representations of an organizational scheme for the ambiguous case. These representations are, from left to right, an organized list, a flowchart, and a table.*



After students presented and discussed their work, I prompted them to reflect on the activity and analyze the pedagogy supporting it. Overall, the students reported enjoying the activity and took satisfaction in developing a deeper understanding of the ambiguous case. They also commented on the openness of the activity and the challenge of developing guiding criteria, organizing their exploration to ensure all cases were considered, and refining their results for clarity and succinctness. Some students remarked that the activity would lend itself to exploration using virtual manipulatives on platforms such as Geogebra (https://www.geogebra.org) or Desmos (https://www.desmos.com). At the conclusion of this discussion, I introduced the concept of representational competence and showed how this activity, with appropriate pedagogy, could support the development of it.

Connections to Representational Competence

Marshall et al. (2010) defined representational competence as "knowing how and when to use particular mathematical representations" (p. 40). Reviewing literature on representational competence, Huinker (2015) summarized four skills that embody the construct:

1. Being able to convey a mathematical idea in various forms;
2. Knowing when and why it is appropriate or valuable to use particular mathematical representations;
3. Being able to translate between or within modes of representation; and
4. Being able to use representations flexibly to solve problems (p. 4).

To facilitate students' development of these skills Marshall et al. (2010) recommended three teaching practices:

- Engaging in dialogue about the explicit connections between representations;
- Alternating directionality of the connections made among representations; and
- Encouraging purposeful selection of representations (p. 40).

I modeled these three practices during implementation of the ambiguous case activity. First, during group presentations, I specifically prompted students to look for specific features across all three representations in Figure 6, such as where the different cases for the measure of $\angle A$ were shown or how the different steps in the decision-making process were shown. In this process, I made sure to phrase my questions so that students' analyses of particular representations were distributed equally, to ensure their reasoning flowed in both directions (e.g., both table-to-flowchart and flowchart-to-table). During our discussion, we also explored the situational advantages and disadvantages of each representation. For example, the table is a more concise and easier to read than the flowchart, but the flowchart helps the user see a specific flow of decisions made during reasoning about a given SSA scenario.

Once the ambiguous case activity had concluded, I introduced the concept of representational competence and Marshall's et al. (2010) three practices. With this new perspective, we analyzed the activity and discussion from a teacher's perspective, considering my decisions and questions during the activity, especially related to using the three practices.

Finally, we considered topics in the secondary curriculum on which purposeful instruction to build representational competence could be implemented. My students identified the ubiquitous use of equations, tables, and graphs to represent functions throughout the high school curriculum as a context in which they could practice representational competence pedagogy on a regular basis. In the context of statistics and data science, they considered the idea of having their students use different visual graphics to represent the same set of data and discuss the affordances of each.

To conclude our discussion, we discussed ways teachers can intentionally plan to implement the three representational competence teaching practices (Marshall et al., 2010). I recommended three steps:

1. Prior to instruction, anticipate useful representations for the topic.
2. Examine the tasks selected for the lesson through the lens of representation, considering whether the tasks collectively have the potential to elicit multiple representations of the topic and encourage bi-directional translations between those representations. Make adjustments as needed.
3. Plan discussion questions focused on the three teaching practices in advance.

These practices, initially performed explicitly, could help teachers internalize the processes needed to support representational competence pedagogy on a regular basis in the practice.

References

Case, C. J. (1989). SSA: The ambiguous case. *Mathematics Teacher, 82*(2), 109–111. https://doi.org/10.5951/MT.82.2.0109.

Cirillo, M., Todd, R., & Obrycki,l J. (2015). *Exploring side-side-angle triangle congruence criterion*. Retrieved from http://udspace.udel.edu/handle/19716/16742 on April 15, 2023.

DeComo, L. S. (2008). The ambiguous case.*Mathematics Teacher, 101*(6), 404–406. https://doi.org/10.5951/MT.101.6.0404.

DeComo, L. S. (2008). Reader reflections: February 2008. *Mathematics Teacher, 101*(6), 404–406. Retrieved Jun 6, 2023, from https://doi.org/10.5951/MT.101.6.0404.

Harrison, E. P. (2002). Using the law of cosines to teach the ambiguous case of the law of sines. *Mathematics Teacher, 95*(2), 114–116. https://doi.org/10.5951/MT.95.2.0114.

Huinker, D. (2015). Representational competence: A renewed focus for classroom practices in mathematics. *Wisconsin Teacher of Mathematics, 67*(2), 4–8. Retrieved from https://s3.amazonaws.com/amo_hub_content/Association1956/files/Journal/WTM%20Spring%202015.pdf on May 18, 2023.

Levine, B. S. (1987). Another approach to the ambiguous case.*Mathematics Teacher, 80*(3), 208–209. https://doi.org/10.5951/MT.80.3.0207.

Levine, B. S. (1987). Sharing teaching ideas. *Mathematics Teacher, 80*(3), 207–212. Retrieved Jun 6, 2023, from https://doi.org/10.5951/MT.80.3.0207.

Marshall, A. M., Superfine, A. C., & Canty, R. S. (2010). Star students make connections. *Teaching Children Mathematics, 17*(1), 38–47. https://doi.org/10.5951/tcm.17.1.0038.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010).*Common Core State Standards for Mathematics* (CCSSO). Washington, DC: Authors. Retrieved from https://learning.ccsso.org/wp-content/uploads/2022/11/Math_Standards1.pdf on May 18, 2023.

Novick, L. R. (2004). Diagram literacy in preservice math teachers, computer science majors, and typical undergraduates: The case of matrices, networks, and hierarchies. *Mathematical Thinking and Learning, 6*(3), 307–342. https://doi.org/10.1207/s15327833mtl0603_3.

Peek, A. L. (1987). Rethinking the ambiguous case.*Mathematics Teacher, 80*(5), 372. https://doi.org/10.5951/MT.80.5.0371.

West, C. (1992). The taming of the ambiguous case.*Mathematics Teacher, 85*(3), 198–200. https://doi.org/10.5951/MT.85.3.0198.

Yeshurun, S., & Kay, D. C. (1983). An improvement on SSA congruence for geometry and trigonometry. *Mathematics Teacher, 76*(5), 364–367. https://doi.org/10.5951/MT.76.5.0364.

Kyle T. Schultz is a Professor of Mathematics Education at the University of Mary Washington in Fredricksburg, Virginia. His work focuses on teachers' decision making with respect to mathematics curriculum, instruction, and technology.