Building a Foundational Understanding of Systems of Equations

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Abstract: *The teaching of systems of equations has often been reduced to a series of procedures and steps, often boiling off the context that is inherent with each individual system. Delivering learning opportunities around systems requires teachers to present students with opportunities to engage with the Standards of Mathematical Practice, where students reason, model, and persevere. In this article, classroom teachers are provided with ideas for practice that center around the use of problems to introduce the various methods for solving simultaneous equations.*

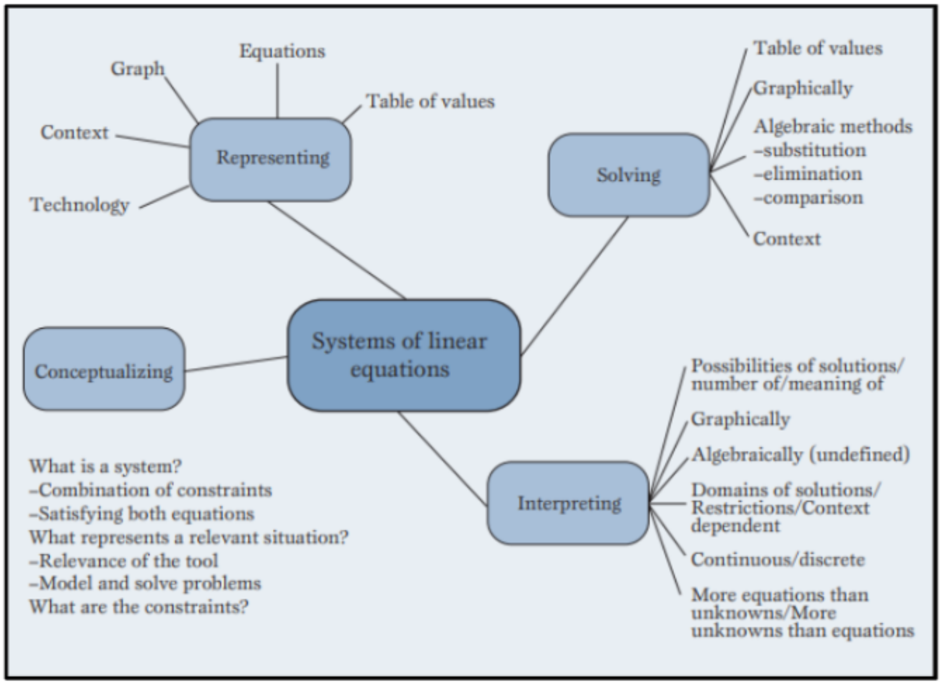
Keywords: *Systems of Equations, Standards of Mathematical Practice, Algebra*

Introduction

Solving systems of equations challenges students to be flexible and mindful of scenarios, structures, and methods that transcend strictly procedural fluency. A system of equations is not a collection of independent equations but rather a set of equations that represents a relationship. A system of equations is a rich mathematical concept used to model, represent, and solve problems in courses ranging from introduction into algebra through calculus (Jerôme et al., 2009). To achieve this level of comprehension, students first must understand the relationship between two variables and what they represent. These relationships can be represented through tables, graphs, and equations. Mastery of these representations establishes the necessary foundation students require to deepen their abstract understanding of systems of equations. Once students can represent and explain a relationship between two variables, they can begin to expand that concept into representing relationships between two or more equations. Correspondingly, Mathematics teachers must understand the power of these representations in order to instruct students adequately.

The concept map portrayed in Figure 1 reflects four areas of focus for learning and teaching systems of linear equations: The meaning of a system of equations, ways of representing a system of equations, ways of solving a system of equations, and interpreting solutions of a system of equations (Jerôme et al., 2009). The concept map gives mathematics educators a visual representation of the connections of multiple ideas related to systems of equations. In the following sections, we discuss ways to solidify students' conceptualization of what a system of equations is and their understanding of how to solve systems. These suggestions provide methods for creating a learning environment that supports the Common Core Standards for Mathematical Practices. Additionally, we will provide an activity to utilize post-instruction as an assessment of students' understanding about how to choose useful solution strategies based on the structure of the system.

Figure 1: *Concept Map of Systems of Equations*



Problems Before Procedures

Before introducing any new topic, assessing students' prior knowledge is vital. To assess students' understanding prior to introducing systems, teachers can present problems that exclude variables and signs. This allows the teacher to observe students' solution strategy, which could be represented by a system of equations or solved in other ways. An example of a Mystery Number problem that appeared in Problems before Procedures: Systems of Equations is provided in Figure 2 (Allen, 2013). Allen suggested presenting students with the problem and allowing them time to think with a partner and share their reasoning. A necessary component of this activity is conducting a classroom discussion in which pairs of students discuss their problem-solving methods. This discussion allows the teacher to gauge the level of prior knowledge that exists among the students.

Figure 2: *Mystery Number Problem*

I'm thinking of two numbers. Their sum is 35, and their difference is 13. What are the numbers?

If students have limited or no exposure to systems of equations, solution methods likely will include a form of guess-and-check (possibly using a table to organize guesses). If this occurs, the teacher can guide the discussion toward representing the relationship between the unknown values as equations using variables. The classroom discussion can begin by discussing what variables are, how they are used, and when they should be used. The goal of the conversation should be for students to realize that a variable is used when a situation has an unknown and identify if the problem at hand indeed has an unknown that can be assigned a variable.

Once the conclusion that variables can be assigned to the mystery numbers has been reached, a discussion should occur about the words in the problem and how these could be translated into symbols that create equations. For example, students need to identify that the word "sum" indicates addition; "difference" indicates subtraction, and "is" is represents "equals" in this problem. It is also important for students to identify the need for two different variables because there are two unknown numbers. If students have significant algebraic knowledge, they may immediately recognize the opportunity to use algebraic equations to represent the Mystery Number problem using variables and solve it using an algebraic method independently.

This activity supports the Standards of Mathematical Practice of Reasoning Abstractly and Quantitatively by calling students to analyze and solve pairs of simultaneous linear equations by understanding that the solution satisfies both algebraic equations (Figure 3) (CCSSO, 2010).

Figure 3: *Standards for Mathematical Practice (CCSSO, 2010)*

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Guided Discovery

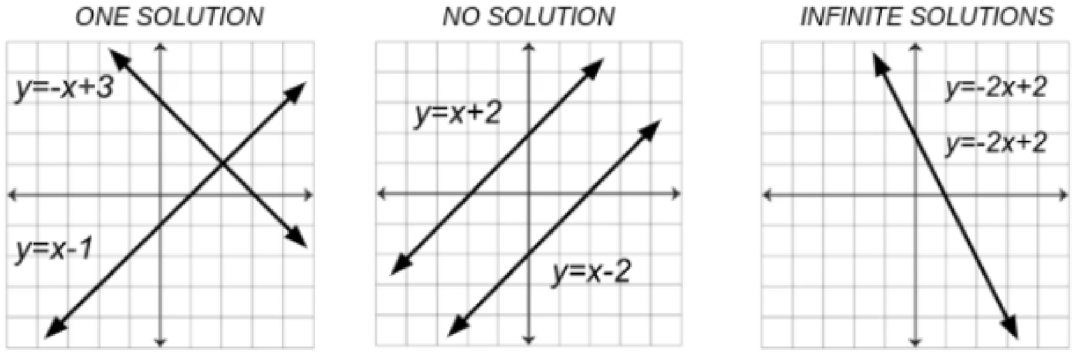
Presenting a problem such as The Mystery Number will activate prior knowledge and begin to build a foundation of understanding. The Mystery Number activity allows the teacher to access the "Representing" domain of the concept map. Students were tasked to use various representations to solve for their mystery numbers. They also were exposed to the obstacle of satisfying two constraints with only one solution which accesses the "Conceptualizing" domain as well. To further build upon this foundation, students should be exposed to similar problems that provide a variety of constraints. Once students grasp the idea of what a system of equations is and more importantly, what its solution represents, different representations of systems such as graphical and algebraic expressions can be introduced.

Guided discovery activities are effective parts of instruction in solving through graphing, substitution, and elimination while enhancing active learning and self-directed learning (Nusantari et al., 2021). Students are given the opportunity to practice critical thinking skills by developing their own processes and applying previous learning rather than memorizing a procedure demonstrated for them. These student-centered activities give students the opportunity to observe, analyze, and draw their own conclusions in order to reach an "Aha!" moment (Gerver et al., 2003). The following lessons are practical tools that can be used to create memorable moments and allow students to construct their knowledge. Guided discovery activities help introduce each method prior to the teacher introducing the concept formally.

Graphing—Number of Solutions

To solve a system of equations by graphing, students can begin with the following activity developed by the authors to discuss observations of the different types of system graphs (Figure 4). This activity can be completed at any point in students' learning regardless of their knowledge about linear equations, as the goal is to make important observations that describe the types of solutions a system of equations can create.

Figure 4: *A Guided Discovery Activity to Determine When Systems Have Zero, One, or Infinite Solutions*



Observations: • • • •	*Observations:* • • • •	*Observations:* • • • •
Conclusions: • *Lines:* • *Equations:*	*Conclusions:* • *Lines:* • *Equations:*	*Conclusions:* • *Lines:* • *Equations:*

This activity can be revisited at any time during students' linear equation journey to reinforce their knowledge of forms and systems of linear equations.

A graphed example of each type of solution set is displayed for students to analyze. Students should be instructed to make observations about the lines and corresponding equations within each example. This step should be student-centered, giving students ample time to write and share their observations. The teacher should prompt the students throughout and guide class discussions to ensure the critical components of each case are identified. From the discussion students should conclude that graphed systems with one solution have one point of intersection and are created by equations with different slopes. Systems with no solution have no point of intersection, creating parallel lines. These equations have the same slope but different y-intercepts. Systems with infinitely many solutions appear to be only one line because two coinciding lines intersect infinitely. Students understand the concept of intersecting and never intersecting (parallel). The most difficult of these concepts for students to visualize and accept is that two different equations can create the same line. In the example provided, students are given two identical equations. To further their understanding of this concept, students could be given equations that do not look the same but create the same graph. It is also important to discuss the term "infinitely many" and what it means in terms of solutions for this system. The students can be prompted to choose a few points on the line to see how they fit into the equations.

If this activity is used before students are familiar with the elements of a linear equation, students still can make observations about how the lines relate to each other and how the equations of the lines differ. In the first image of Figure 4, students can observe that x is positive in one equation but negative in another. These lines intersect. In the second graph, the x in both equations is positive, and the lines are not intersecting. The terms "slope" and "intercept" can be introduced later.

Substitution

The substitution method is one of two algebraic methods used to solve systems. Prior to formalizing the procedures of this method, students will benefit from participating in a guided discovery introduction (Figure 5).

Figure 5: *A Guided Discovery Introduction to Substitution Method*

System 1:	Questions:
x+y=4	1) How is the second equation helpful in determining the value of x?
y=1	2) How can the values for each variable be checked?
System 2:	
x+y=12	3) What is the result when y is replaced with 2x-3 in the first equation?
y=2x-3	4) Why is this allowed?
	5) What is the value of x?
	6) How can the value of x be helpful in finding the value of y?
	7) Do the values for x and y make both equations true?

In this short activity, students answer questions that give them the opportunity to analyze the steps in the substitution method. The first system provides the numerical value of y that can be substituted into the first equation to find the value of x. This form of substitution is introduced in evaluating expressions, and therefore, students relate their prior knowledge to this new form of substitution.

While discussing this system, students should observe that the y in the first equation is eliminated when replaced with 1, creating a one-variable, two-step equation. In the second system, an algebraic expression is equivalent to y.

Following the same procedure as the first system, y is replaced with its equivalent expression in the first equation again creating a one-variable equation. It is essential that the teacher facilitates the discussion and allows students to generate a procedure by comparing the similar process used in both systems. Teachers should ask students what they see as the same in both systems or how this process reminds them of problems they have done before.

Students may compare the similarities of the two problems in terms of how the substitution is performed. Teachers should guide the students toward the discovery that the process essentially is the same in both systems and the only difference is substituting an expression rather than a single number.

Guided discovery also can be used to introduce the elimination method. This activity will allow students to understand that like terms in different equations can be added (or subtracted). To begin, the teacher presents a few numerical equations that students can verify to be true (Figure 6).

Figure 6: *Using Guided Discovery to Determine That Like Terms in Equations Can Be Added and that When Opposites are Added, a Term is Eliminated*

$$\begin{array}{r} 3+4=7 \\ +\;1+2=3 \\ \hline = \end{array} \qquad \begin{array}{r} 2+6=8 \\ +\;5+9=14 \\ \hline = \end{array} \qquad \begin{array}{r} 2x+y=5 \\ +\;4x-y=7 \\ \hline = \end{array}$$

Then, students are prompted to add the verified equations together vertically and analyze the resulting equation. Students then verify the resulting numerical equation to be true also. For instance, the result of adding the terms in the first example is $4 + 6 = 10$. The process is repeated with the second example. Once students understand that this is possible with numerical equations, they are given algebraic equations to complete the same process. Students follow the same process with the third example as they did with the first two. The teacher asks questions during this process to guide their thinking such as "What happens if you add the corresponding parts of each equation?"

Following the activity, teachers and students should discuss what happened when the like terms were added to emphasize that adding opposite coefficients of the y terms caused those terms to be eliminated in the resulting equation. Students are prompted to think about how having an equation with only one variable could be helpful in solving the system. Once the value of x is found, the discussion can focus on how that x value can be used to find the y-value.

To further the students' understanding, we follow up this example with a system of equations that requires students to eliminate a variable with coefficients (Example 1 in Figure 7).

Figure 7: *Extended learning of the Elimination Method*

Example 1:

$$-2x + 3y = 12$$
$$2x - 4y = -3$$

Example 2:

$$-7x - 8y = 9$$
$$-4x + 9y = -22$$

Example 3:

$$4x + 2y = 12$$
$$4x + 8y = 24$$

Students will discover the need for both coefficients to have the same absolute value while having opposite signs in order to be eliminated. As they advance even further into learning this concept, students can be given a system in which neither variable has the same absolute value (Example 2 in Figure 7) and a system where the variables have the same sign (Example 3 in Figure 7). This activity lays the foundation of what causes a variable to eliminate, allowing students to solve for the remaining variable. Once this understanding is established, students have the ability to take on more advanced structures.

Elimination becomes increasingly important as students advance through to higher-level courses that form that basis of calculus that introduce systems with three or more unknowns. Students can initially be taught these systems using a multiple elimination method. Students should explore these examples on their own using their previously learned elimination and substitution skills. The goal in two equation systems is to eliminate one unknown in order to solve for the other. In this case, the initial goal will be the same—eliminating one of the unknowns in all three equations.

The teacher should guide the students towards the idea that elimination can be done more than once, as students will quickly discover it is not possible by only performing one time. The process of performing elimination multiple times will create another system of two unknowns that can be solved in the traditional methods previously discussed. As students become fluent in the process of dealing with more than two unknowns, teachers can introduce an even more advanced process of solving systems using matrices.

While transcending the trajectory of the current work, Cramer's Rule uses the determinants of matrices to solve systems of equations and stands as an excellent way to differentiate instruction across a wide range of learners, while also providing a platform to robustly explore systems beyond two variables.

Selecting Systems of Linear Equations

As students become familiar with these three solution strategies, they can begin to discover how the methods are used and when each method can be used appropriately. A student may prefer one method over another, but it is important also to acknowledge that methods can be chosen based on the structure of every given system. Selecting which method to use to solve the system most efficiently can be difficult for students. The goal is to equip students with multiple strategies for thinking about and solving systems, including the practice of noticing characteristics of systems that might suggest specific solution strategies.

Otten and Otten (2016) created an activity that allows students to think about useful strategies for solving systems of linear equations. First, students should familiarize themselves with the equations, verifying that they all are linear equations written in different forms (Figure 8).

Figure 8: *List of the Eight Equations*

Below is a set of eight equations. Take a moment to verify that they are linear (How do you know?) and to notice the different forms of the equations (e.g., slope-intercept form, general form, simplified, not simplified).

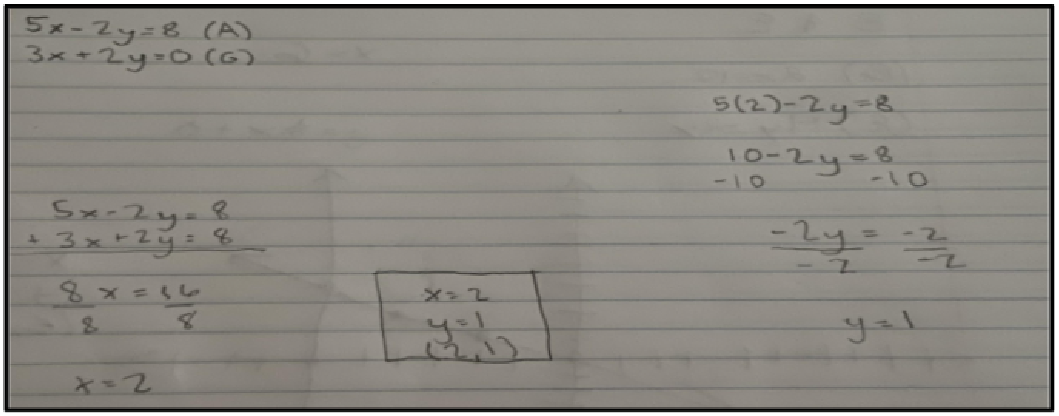
A)	$5x - 2y = 8$	E)	$4y = 2x$
B)	$2x = 12$	F)	$y = -x/2 + 2$
C)	$-2x + 4y = 12$	G)	$3x + 2y = 0$
D)	$y = -3$	H)	$9x + 6y = 0$

Teachers can guide the students through a discussion about the different forms of linear equations they see in the list. To complete the activity, pairs of students choose two equations from the list to form systems that they will solve using one of the three methods: substitution, elimination, or graphing. As students solve the systems, they are required to reflect on their choices of equations and solution strategies. Encouraging students to think about their choice enables teachers to assess whether the students have a foundational understanding of the concepts taught within the unit. These decisions can lead to discussions about mathematical structures and relationships as students defend or amend their choices.

Research Findings: Student Work Samples

We gave our activity to a sample of students as research for this article, and the results are described as follows. One group of students chose equations A and G to practice the elimination method. This is illustrated in Figure 9. The students discussed why they chose these two equations to practice elimination. One student said, "both equations line up, which set it up to use elimination."

Figure 9: *Sample Student Work Highlighting the Elimination Method*

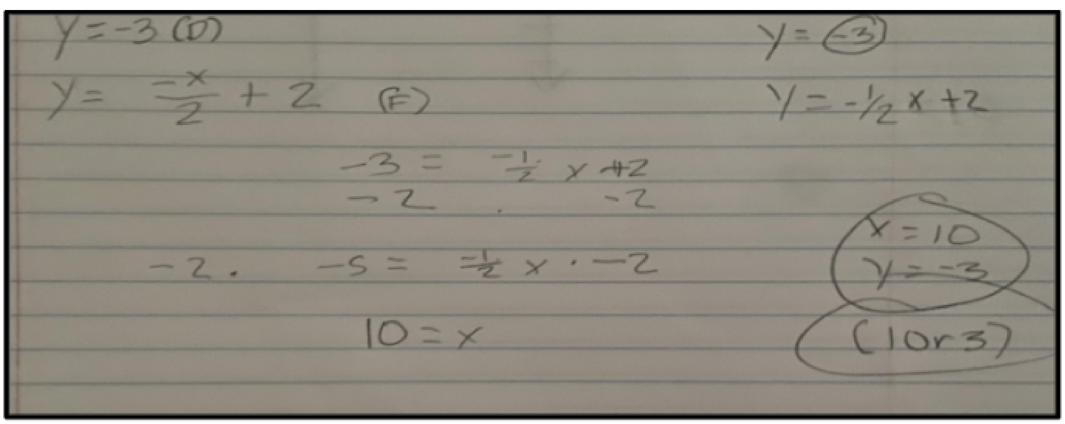


It is possible the students used “line up” to identify that both equations were in standard form, and therefore, the x and y terms were in the same position when written vertically. Along with lining up, students expressed that the coefficients in front of y also helped them choose these equations.

When looking for a system for which they could use elimination, the students realized that the coefficients for y were opposites (2 and -2). To solve the system, the students followed the process of adding the two equations together, ultimately eliminating the variable y and allowing them to solve for x. The students understood that they were solving for an ordered pair and thus used the x-value to perform substitution and find the value of y.

Another group of students decided to use the substitution method with equations D and F. Their work is highlighted in Figure 10.

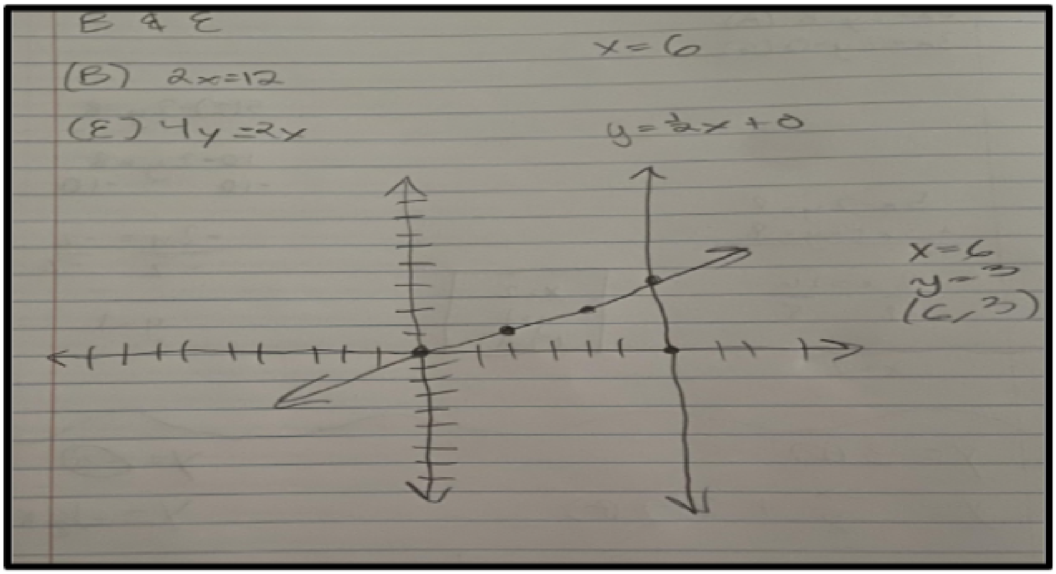
Figure 10: *Sample Student Work Highlighting the Substitution Method*



When discussing why the group chose these two equations for substitution, one student explained, “For D, y was already by itself. When it was by itself, it made it easy to plug in the equation and solve for x.” The students chose to use equation F, assuming that the expression for y in the first equation could be substituted into any equation with a y variable. Equation D provided the students with the value of y and allowed them to recognize they would replace the variable y in Equation F with that value. They correctly replaced y with -3 and solved for the value of x, successfully finding the solution of the system.

A third group used graphing as their method of choice, as shown in Figure 11.

Figure 11: *Sample Student Work Highlighting the Graphing Method*



After some discussion, the group chose equations B and E because the students felt they would be the easiest to graph. "We just need x and y by itself." The students' explanation demonstrated their understanding of graphing equations when presented in different forms.

The students summarized their thought process, noting that Equation E can be simplified into slope-intercept form, while Equation B is a single variable equation that can be further simplified using division.

We discussed what it means to have a single variable equation and what that line may look like. The students reasoned that an equation containing only an x-variable will have that same x-value for each y value on the graph—noting that Equation B would generate a vertical line.

This group did not have graph paper provided for them, so they drew their coordinate plane for the activity. After graphing the two lines using our discussion of vertical lines and their knowledge of slope-intercept form, they found the intersection point on the graph.

Whole Group Discussion

The class discussed the different ways in which these problems can be solved. The students pointed out that all three strategies should result in the same solutions for a given system (if there are no errors), recognizing that graphing may not always produce a precise solution. This activity led to engaging discussions between the groups of students. As there were many debates between the groups about the "easier ways" to solve the systems, each group was able to discuss why they picked the equations and solved the systems the way they did.

This activity facilitates substantial discussion in a mathematics classroom. Students are given the opportunity to explain their thought processes and reasoning behind their decisions. The activity also can be expanded to have students solve their chosen system using all three methods discussed in this article. This extension allows the students to consider which method was the most efficient for a particular system and identify the structures of the equations that make that method the easiest. At the end of the activity, students should have a foundational understanding of the concept that multiple methods can be used to solve systems, but the structures of the equations will make a difference in the efficiency of each method.

Complexity of Coefficients

An important consideration when introducing all methods of solving systems of equations is to increase the complexity of examples. Students should be exposed to the variety of equations they may see when given a system of equations. Each of the examples introduced throughout this article include integer coefficients only. A simple way of increasing the complexity of these examples is by using rational coefficients. Rational coefficients become more difficult when choosing elimination or substitution. Students do have a choice in how they approach these coefficients and therefore, should be presented with a problem and given the opportunity to explore and justify viable problem solving methods. Many students may choose to work with the rational numbers, creating more complexity and room for error when performing the algebraic steps to solving the system. If this is the case, teachers should prompt student thinking of what is known about rational numbers and what operations can be performed with them. Discourse and justification is essential, student-student, teacher-student, student-teacher. It is important that these discussions ensure students have a sound foundation of how to maintain equality.

Conclusion

To help students develop a rich understanding of systems of equations, we suggest presenting problems before procedures and guided discovery activities prior to formalizing each solution method. Following experience with each method, facilitating the selecting systems activity will allow students to apply their understanding through the decision-making process which utilizes the Standards of Mathematical Practices of (1) making sense of problems and persevering in solving them; (2) reasoning abstractly and quantitatively; and (3) modeling with mathematics. These activities provide teachers with the tools to help students develop a solid foundational understanding of systems of equations.

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