Formulas! Formulas! Everywhere!

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Abstract: *The authors describe activities that can be used to introduce the concepts of perimeter and area with conceptual understanding for students in grades 4 through 9. The activities use a transformations-based approach (e.g., reflection and rotation) to build students' conceptual understanding.*

Keywords: *area, geometry, measurement, transformation*

Introduction

Throughout the years of teaching high school and college, it frequently happens that students do not know the origin of the area formulas for many common geometric shapes. Typically, the students are told the formula during their education, what each variable represents, substitute numbers into the formula and then evaluate. Evidently, there has been no algebraic or geometric development of the formulas. What is proposed here is a series of activities that start to build the conceptual basis of area and a geometric development of the formulas. Since these activities span a range of grade levels, feel free to pick and choose which activities you use in your classroom. The grade level as described in the Common Core State Standards Initiative for Mathematics is found in a table with the activity at the end of the article.

The following activities were designed to fulfill (at least in part) the above recommendations with respect to the concepts of symmetry and area. In addition, these activities address Model with Mathematics, a Standard for Mathematical Practice, which states that mathematically proficient students should "routinely interpret their mathematical results in the context of the situation (CCSSI 2010). The activities also address Make Sense of Problems and Persevere in Solving Them because students must "start by explaining to themselves the meaning of a problem and looking for entry points to its solution (CCSSI 2010).

According to the NCTM's Principles and Standards document (NCTM 2000), students in grades 3-8 should:

- Understand measurable attributes of objects and the units, systems, and processes of measurement.
- Apply appropriate techniques, tools, and formulas to determine measurements.
- Use visualizations, spatial reasoning, and geometric modeling to solve problems.

In addition, the Common Core State Standards (Corestandards.org, 2010) recommend that all students:

- Understand and use concepts of area.

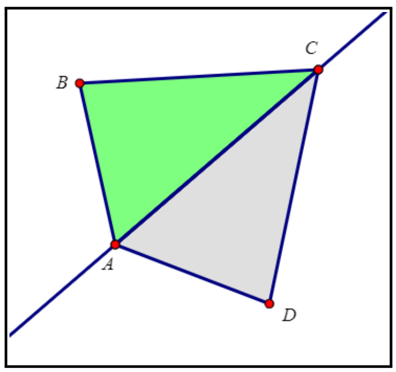
- Recognize perimeter as an attribute of plane figures and distinguish between linear and area measures.
- Draw, construct and describe geometrical figures and describe the relationships among them.
- Recognize lines of symmetry for two-dimensional figures. Identify figures that have lines of symmetry and draw lines of symmetry.

Activities with Transformations: Reflections

Begin with a review or lesson about reflecting or rotating figures. (See Krach and Moyer 2020 for some ideas.) Draw an acute scalene triangle on patty paper. Fold and crease it along any side of the triangle. See Figure 1. What quadrilateral has been created? Ask the students questions like "What do you notice?" and "What do you wonder?"

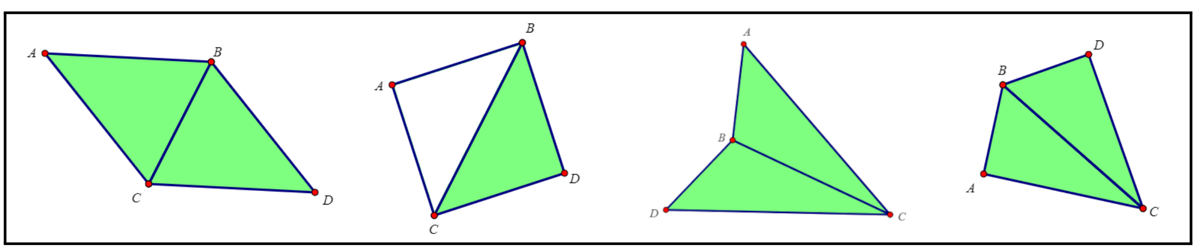
Most students typically mention how they had never seen quadrilaterals in this manner, nor quadrilaterals as an extension of transformations.

Figure 1: *Reflecting a triangle over one of its sides.*



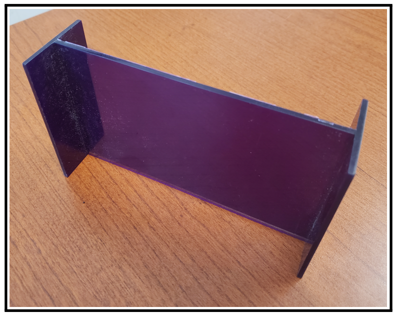
Since a reflection preserves distance and angle measure, it will create a congruent triangle with sides $\overline{AB} \cong \overline{AD}$ and $\overline{BC} \cong \overline{DC}$. The resulting quadrilateral is a kite. On separate sheets of patty paper, ask students to reflect an isosceles triangle, an isosceles right triangle, an obtuse triangle and a non-isosceles right triangle. Continue to ask, "What do you notice?" and "What do you wonder?" These reflections should result in a rhombus, a square, a dart and a kite respectively, as in Figure 2.

Figure 2: *Reflecting a triangle over one of its sides.*



An alternate way to perform reflections includes a Mira (Figure 3), a tool that acts as a mirror, which is especially useful for elementary school students. The drawing edge of a Mira is beveled. This edge should be placed on the line of reflection and facing the user. Look directly through the Mira to locate the image of the object on the other side of the Mira. A Mira can also be used to determine the line of reflection. If a Mira is placed between an object and its reflection, move the Mira until the object falls completely on the reflection.

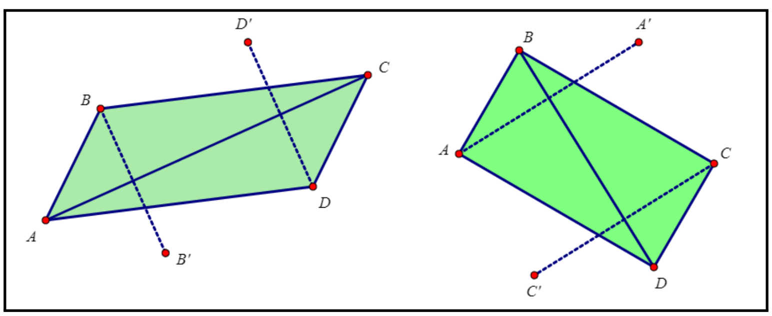
Figure 3: *A Mira.*



Another alternate method would be the use of WebSketchpad (WSP) for this investigation. Found at https://geometricfunctions.org/fc/tools/library/, WSP is an intuitive dynamic geometry software package where students can easily construct and investigate geometric concepts to observe patterns. WSP allows students to drag one point of a triangle that has been reflected over any side to see what type of quadrilateral is constructed. In addition to a Mira, WSP can also be used to determine which quadrilaterals cannot be constructed through reflection. This compare-and-contrast activity will lead into a discussion about shapes with reflective symmetry.

Rhombi, squares, kites and darts have reflective symmetry over either diagonal (refer to Figure 2) whereas Figure 4 illustrates that parallelograms and rectangles (that are not squares) do not. A rectangle and an isosceles trapezoid do have reflective symmetry over certain midsegments. What shapes are needed to create the rectangle and isosceles trapezoid? Can a parallelogram be created in some manner?

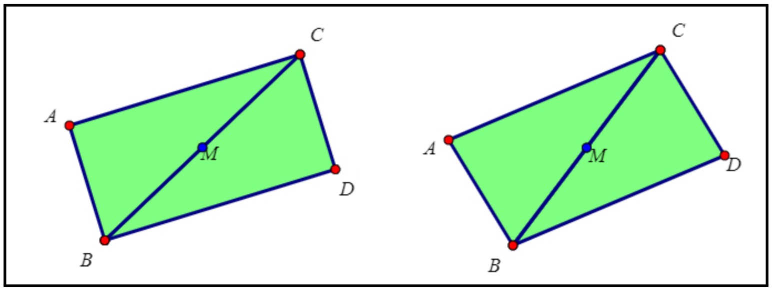
Figure 4: *Parallelograms and rectangles have no reflective symmetry.*



Activities with Transformations: Rotations

Ask students to construct any triangle and locate a midpoint on any one of the three sides. Have them trace the triangle onto patty paper. With the tracing over the original triangle, place the pencil on the midpoint and rotate the patty paper 180 degrees. See Figure 5. What shape has been created? What if the original triangle was a scalene right triangle? Any triangle rotated over a midpoint will create a parallelogram where a scalene right triangle rotated over the midpoint of the hypotenuse will yield a rectangle. Ask the students "Why does this happen?" These reflections and rotations will be revisited when developing the area formulas.

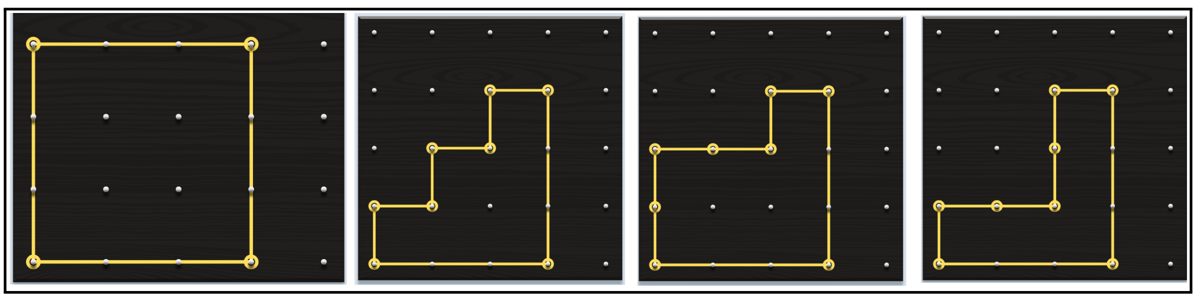
Figure 5: *Results of rotating a triangle about a midpoint of a side.*



A Side Trip for a Misconception about Perimeter and Area

Though this paper is focusing on the area formulas, we feel that one related misconception that should be treated here is the relationship between perimeter and area. Consider the four figures in Figure 6. If the horizontal and vertical distance between pegs is accepted as being the unit length, ask the students to compute the perimeter of each shape and ask them what they notice.

Figure 6: *Four perimeter problems.*

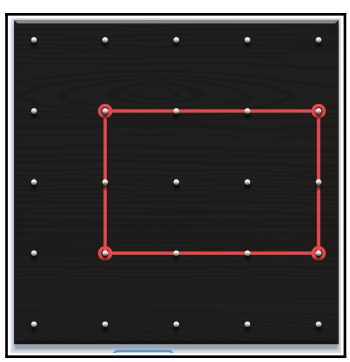


The students should find that each figure has a perimeter of 12 units. These shapes will be revisited later after an introduction to computing area.

Basic Area – Counting Squares

For the basic concept of area, start with a geoboard. There are free geoboards available on online, such as at the Math Learning Center (https://apps.mathlearningcenter.org/geoboard/).

Figure 7: *Counting squares.*



Ask students to count the number of squares on the geoboard as shown in Figure 7. This figure has an area of 6 squares. Continue with several examples for students to develop the formula for a rectangle as the product of the lengths of the sides. It should be emphasized repeatedly that the lengths of the sides must be perpendicular to each other since the basic shape or unit being counted is a square.

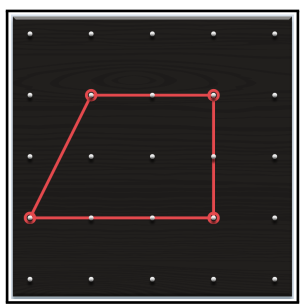
The Misconception Returns

Revisit the four figures in Figure 6 to find their areas. The areas are 9 squares, 6 squares, 7 squares and 5 squares, respectively. Ask the question "If two shapes have the same perimeter, must those shapes have the same area?" These examples illustrate that the answer is no. But what if two shapes have the same area? Must those shapes have the same perimeter? On dot paper, ask students to create as many rectangles as possible that have an area of 24 squares. (The rectangles have dimensions of 1×24, 2×12, 3×8 and 4×6.) Ask the students what they notice about the perimeters. Similar to the previous activity, two shapes that have the same area do not necessarily have the same perimeter. Putting these two observations together, there is no direct relationship between perimeter and area of two shapes.

Activities with Area: The Beginning of Formulas – Pick's Formula

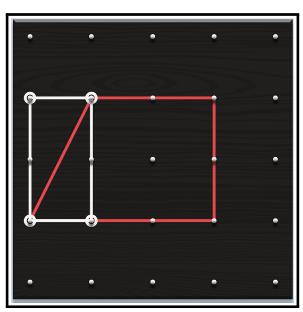
Challenge the students with an example like the following in Figure 8.

Figure 8: *Find the area.*



The area of the shape is at least 4 squares since there are four complete squares in the middle and third column of the shape. Ask students how they might determine the number of squares contained within the left side of the figure. See if students construct a rectangle as shown below (Figure 9).

Figure 9: *Rectangle created.*



Notice that the white rubber band has created a rectangle of area 2 squares. The two triangles formed by the diagonal are congruent through rotation, only half of that area is needed. Therefore, the red shape has an area of 5 squares. This example introduces the idea of taking half of the area of a figure making up two congruent pieces.

Now find the area of shape 1 shown below in Figure 10. Constructing rectangles will give an area of 6 squares. Have the students verify this area. Now ask the students to count the number of border pins and interior pins. Record these numbers along with the area in a table (see Table 1). Challenge the students to look for a pattern in the numbers of each pin with the area of that shape.

Figure 10: *Rectangle created.*

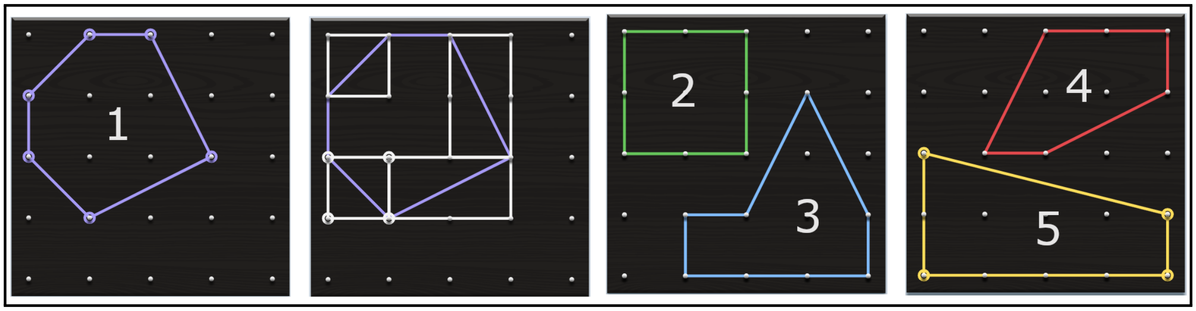
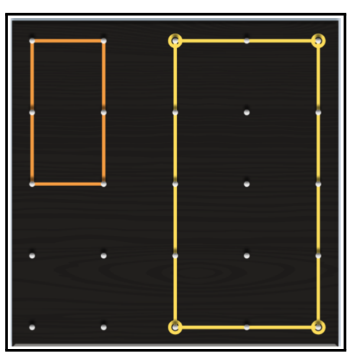


Table 1: *Number of pins.*

Shape	Purple 1	Green 2	Blue 3	Red 4	Yellow 5
Border pins	6	8	8	6	8
Interior pins	4	1	2	2	3
Area (in square units)	6	4	5	4	6

Hopefully, students will see the pattern known as Pick's Theorem if given enough time for discussion. Challenge the students to create a formula that relates the area of the figure to the number of border pins and interior pins.[1] This activity demonstrates the convenience of using a formula. Instead of counting squares by possibly constructing rectangles to determine an area of a figure, using a formula allows a student to find the value of one or two attributes of that figure to calculate the area. In this case, use the attributes of border pins and interior pins to calculate the area. This activity also leads into the reason why the units of area measurement are called square units. The authors will use Pick's Theorem to develop the area formula for a rectangle (Figure 11).

Figure 11: *Two rectangles.*



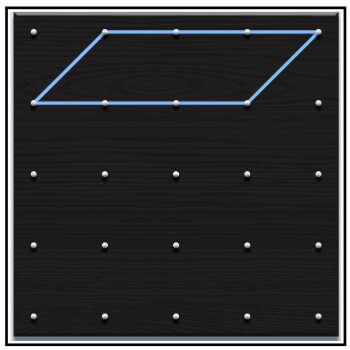
[1]Pick's Theorem: Area $= \frac{1}{2}B + I - 1$ where B is the number of border pins and I is the number of interior pins.

Activities with Area: Developing Common Area Formulas

Using Pick's Formula, the area of each rectangle in Figure 11 can be found to be 2 and 8 square units respectively. Challenge the students to make the connection between the products of the lengths of the two sides of the rectangle with its area to establish the formula. For consistency, it is recommended that the formula $Area = base \times height$ be used, where the base and height of the figure must be perpendicular to each other. Since a rectangle has four right angles, it does not matter which side is chosen to be a base or a height. Since every square is also a rectangle, the area formula for a square will also be $Area = base \times height$.

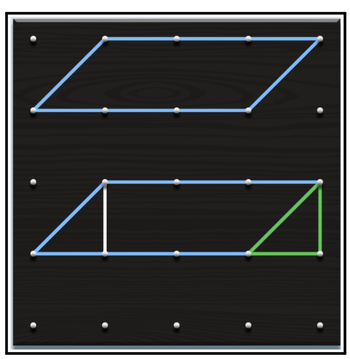
Construct a parallelogram on a geoboard as shown in Figure 12. Since the sides are no longer perpendicular, the product of the two sides cannot be used to calculate the area of the figure. This is the reason it is not recommended using length times width for the formula for the area of a rectangle. Now construct a perpendicular from one of the vertices of the parallelogram to a side (the white rubber band). See the white rubber band in Figure 13.

Figure 12: *A parallelogram to find area.*



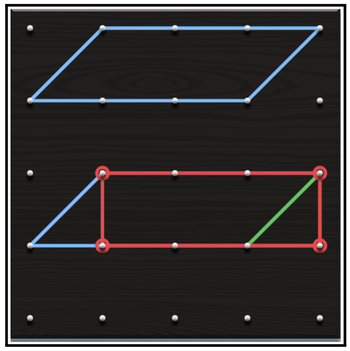
Now translate this newly formed green triangle to the other end of the parallelogram (see Figure 13). What do you notice about the triangle on the left (with the white rubber band) and the triangle on the right (with the green rubber band)?

Figure 13: *Translating a triangle.*



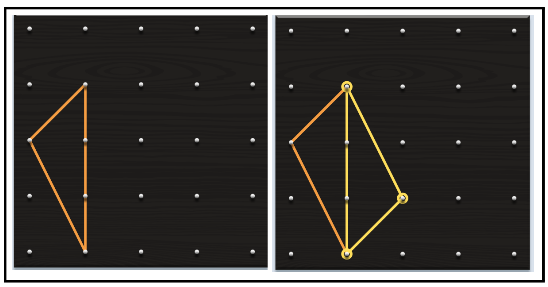
Using Figure 14, ask the students "What must be true about the area of the original blue parallelogram and the rectangle outlined with the red rubber band?"

Figure 14: *Comparing a parallelogram to a rectangle.*



Now we will turn the students' attention to a triangle as in Figure 15. Make a triangle such as the orange one shown on the geoboard. How can we make it a parallelogram? Recall from an earlier discussion that a reflection did not create a parallelogram, but a rotation did. Rotating the triangle about a midpoint is shown with the yellow rubber band and results in a parallelogram.

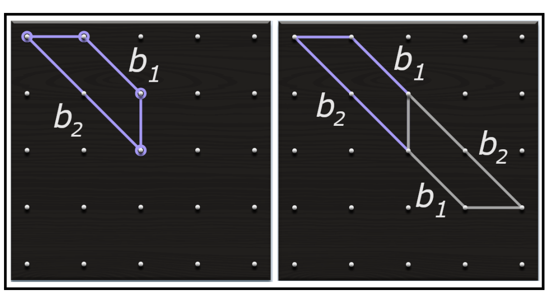
Figure 15: *Rotating a triangle to form a parallelogram.*



From the previous discussion (Figure 14), the area formula for a parallelogram is $A = Bh$, where B represents the length of the base and h represents the length of the height with $b \perp h$ (b is perpendicular to h). Here, only half of the parallelogram is needed so the area formula for a triangle would be $A = \frac{1}{2}bh$ with $b \perp h$.

In Figure 16, the formula for a trapezoid can be demonstrated in the same manner. Start with any trapezoid on a geoboard. Rotating the trapezoid about the midpoint of one of the legs creates the gray trapezoid. Overall, the shape is now a parallelogram with a base of length $b_1 + b_2$. The area of the parallelogram can be calculated by using $A = (b_1 + b_2)h$. Again, only half of the parallelogram is needed so the formula becomes $A = \frac{1}{2}(b_1 + b_2)h$.

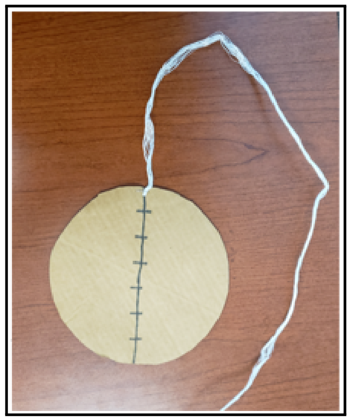
Figure 16: *Rotating a trapezoid about a midpoint.*



Activities with Area: Formulas for Circles

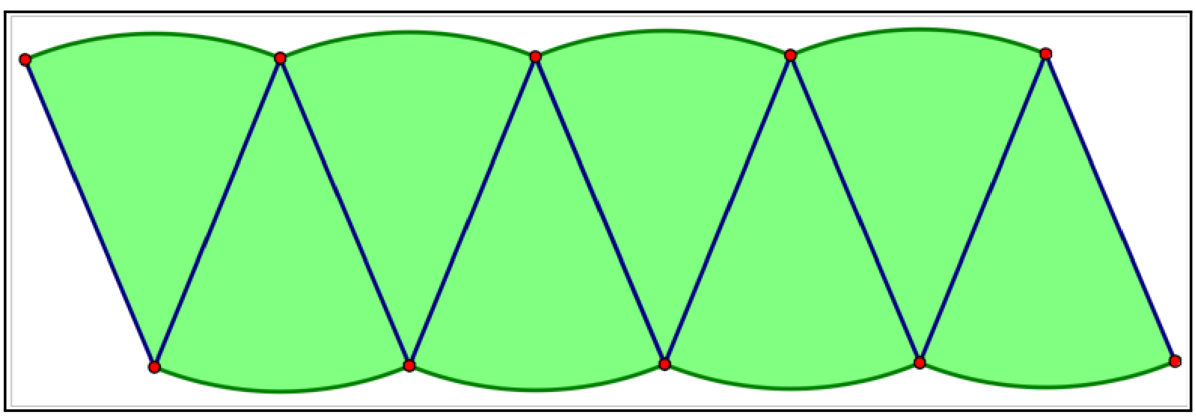
Before the area formula for a circle is developed, there is a need to develop the formula for circumference. Construct a circle on a piece of cardboard and cut it out as shown in Figure 17. Draw a diameter. Divide the diameter into seven equal parts. Attach a string to one end of the diameter. Carefully wrap the string around the circle and hold it where it laps the circle once. Now wrap that string around the diameter and count the number of diameters contained within the string. The string should make three complete diameters and have approximately one-seventh of a diameter leftover: $\frac{C}{d} \approx 3\frac{1}{7} \to \frac{C}{d} = \pi \to C = \pi d$. However, the preferred formula is $C = 2\pi$, since a diameter consists of two radii $d = 2r$.

Figure 17: *A tool for circumference.*



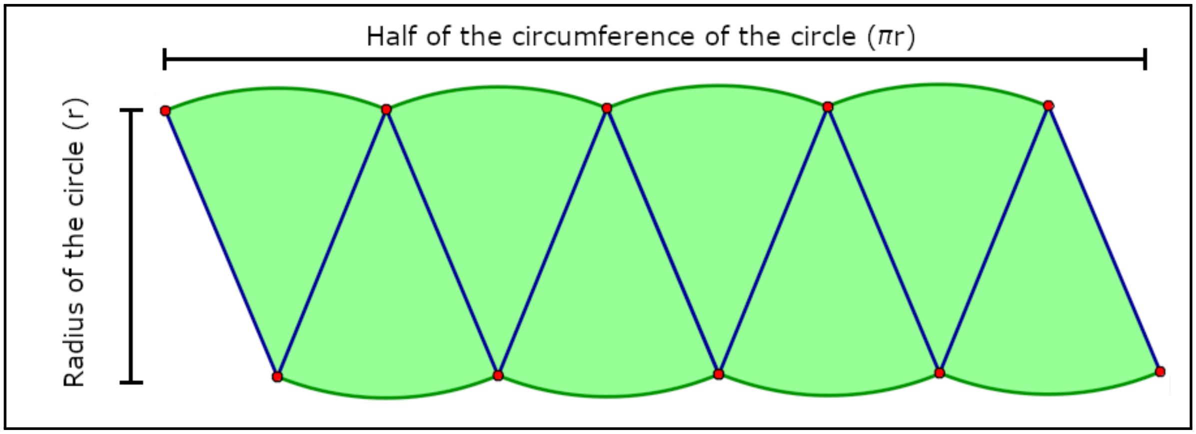
For the formula for the area of a circle, start with a previously drawn circle on paper or an actual paper plate. With paper, ask the students to cut out the circle and fold it in half, then half again, and then in half one more time. Each fold should contain the center of the circle on the crease and the last fold should result in a cone. Using the paper plate as an alternative, fold the plate also in half three times, in the same manner as the paper circle. Regardless of method, have the students unfold the circle and cut out the eight sectors as shown in Figure 18.

Figure 18: *Arrangement of eight sectors.*



What "quadrilateral" has been created here? Challenge the students to determine its base and height. Since the shape resembles a parallelogram, see Figure 19 for its dimensions.

Figure 19: *Dimensions of the "parallelogram."*



Since the area of a parallelogram is found by multiplying the base and height of the parallelogram, this area can be found by the product of half of the circumference and the radius: Area $= \pi r \times r = \pi r^2$. (For additional area activities with triangles and other polygons, see Krach 2015 in the Reference section.)

Conclusion

Over the years, many of our students have indicated that the area formulas of most common geometric figures have never been fully developed for them. An introduction and presentation of the formulas followed by a few specific examples was the extent of the "teaching" and, consequently, the "learning" of the formulas. This type of activity provided a form of reasoning as to what the formulas were based upon and as to the reason the formulas "worked". Although this is not a valid replacement for the formal proof of the formulas, this reasoning is sufficient as a basis from which to build the algebraic proof of each formula.

As a formative assessment, ask students to complete an Exit Pass (see Appendix A for sample exit passes that we have used with school-aged and college-level students). It is important for students to have a venue where they are permitted/required to summarize and communicate their learning (SMP #3 - Students justify their thoughts, share them with others, and critique the responses of others). In addition, any information included in the exit passes should be used to determine the content of the next day's lesson (i.e., review, continuation or extension). When you try these activities with your students, contact the authors with any comments/suggestions/questions based on your students' responses and comments.

Refer to Tables 2 and 3 for suggestions regarding grade level activities and appropriate CCSS-M.

Table 2: *Recommended activities and CCSS-M alignment.*

Level	CCSS-M Content	Activity
Grade 1	CCSS.MATH.CONTENT.1.G.A.2 Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.	Activities with Transformations: Reflections Activities with Transformations: Rotations
	CCSS.MATH.CONTENT.1.G.A.3 Partition circles and rectangles into two and four equal shares, describe the shares using the words *halves*, *fourths*, and *quarters*, and use the phrases *half of*, *fourth of*, and *quarter of*. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.	Activities with Area: Formulas for Circles
Grade 2	CCSS.MATH.CONTENT.2.G.A.2 Partition a rectangle into rows and columns of same-size squares and count to find the total number of them.	Basic Area: Counting Squares
	CCSS.MATH.CONTENT.2.G.A.3 Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.	Activities with Area: Formulas for Circles
Grade 3	CCSS.MATH.CONTENT.3.G.A.1 Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.	Activities with Transformations: Reflections Activities with Transformations: Rotations
	CCSS.MATH.CONTENT.3.MD.C.5 Recognize area as an attribute of plane figures and understand concepts of area measurement. CCSS.MATH.CONTENT.3.MD.C.5.A A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area. CCSS.MATH.CONTENT.3.MD.C.5.B A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units. CCSS.MATH.CONTENT.3.MD.C.6 Measure areas by counting unit squares (square cm, square m, square in, square ft, and improvised units). CCSS.MATH.CONTENT.3.MD.C.7 Relate area to the operations of multiplication and addition.	Activities with Area: The Beginning of Formula – Pick's Formula Activities with Area: Developing Common Area Formulas

Table 3: *Recommended activities and CCSS-M alignment (continued).*

Level	CCSS-M Content	Activity
Grade 3	CCSS.MATH.CONTENT.3.MD.D.8 Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters.	A Side Trip for a Misconception about Perimeter and Area
	CCSS.MATH.CONTENT.3.G.A.2 Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. *For example, partition a shape into 4 parts with equal area, and describe the area of each part as 1/4 of the area of the shape*.	Activities with Area: Formulas for Circles
Grade 4	CCSS.MATH.CONTENT.4.G.A.3 Recognize a line of symmetry for a two-dimensional figure as a line across the figure such that the figure can be folded along the line into matching parts. Identify line-symmetric figures and draw lines of symmetry.	Activities with Area: The Beginning of Formula – Pick's Formula Activities with Area: Developing Common Area Formulas
Grade 5	CCSS.MATH.CONTENT.5.G.B.3 Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles. CCSS.MATH.CONTENT.5.G.B.4 Classify two-dimensional figures in a hierarchy based on properties.	Activities with Transformations: Reflections Activities with Transformations: Rotations

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Appendix A: Samples of Exit Passes

Exit Pass Sample 1

1. Please submit your responses to these questions before exiting this class. Name one mathematical concept or skill that you learned or "revisited" in today's class. Be specific.
2. List one positive thing that happened in this class today (with respect to the lesson). Explain why you felt that it was positive. Please be specific.
3. List one thing that happened in class today that you would change (with respect to the lesson). Explain why you would change it. Please be specific.

Exit Pass Sample 2

I am Noticing	I am Wondering

Exit Pass Sample 3

I used to think . . .

Now I know . . .