Exploring Hyperbolic Geometry (Part 2): Angles and Triangles

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Abstract
This article is the second of a series in investigating basic concepts of hyperbolic geometry using WebSketchpad as the vehicle for the investigations. This particular article focuses on angle relationships and theorems about triangles.

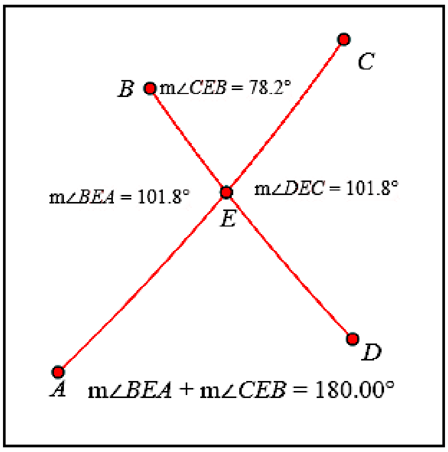
Keywords: Hyperbolic Geometry, WebSketchpad, Proof and Argumentation

1 Introduction

In the first article of this series, we examined some of the history of hyperbolic geometry and from where it grew. The most significant difference between hyperbolic and Euclidean geometries is that there is more than one line parallel to a given line through an external point. The existence of these multiple parallel lines causes some changes to what was studied in Euclidean geometry. The largest effect is that triangles no longer have an angle sum of exactly 180˚, now being strictly less than 180. This also implies that a quadrilateral sum is less than 360°, thus denying the existence of rectangles in hyperbolic geometry. This article will consider the effects on angle relationships and properties of triangles.

In hyperbolic geometry, are two angles of a linear pair still supplementary? *Are a pair of vertical angles still congruent?* A quick construction in WSP with measures should be enough to confirm that both facts remain true, just as they were in Euclidean geometry. (Note: the Poincaré disk may not be shown on all remaining sketches.) The proofs of these two theorems are exactly the same as in Euclidean geometry.

Figure 1
An example of linear pairs and vertical angles in hyperbolic geometry

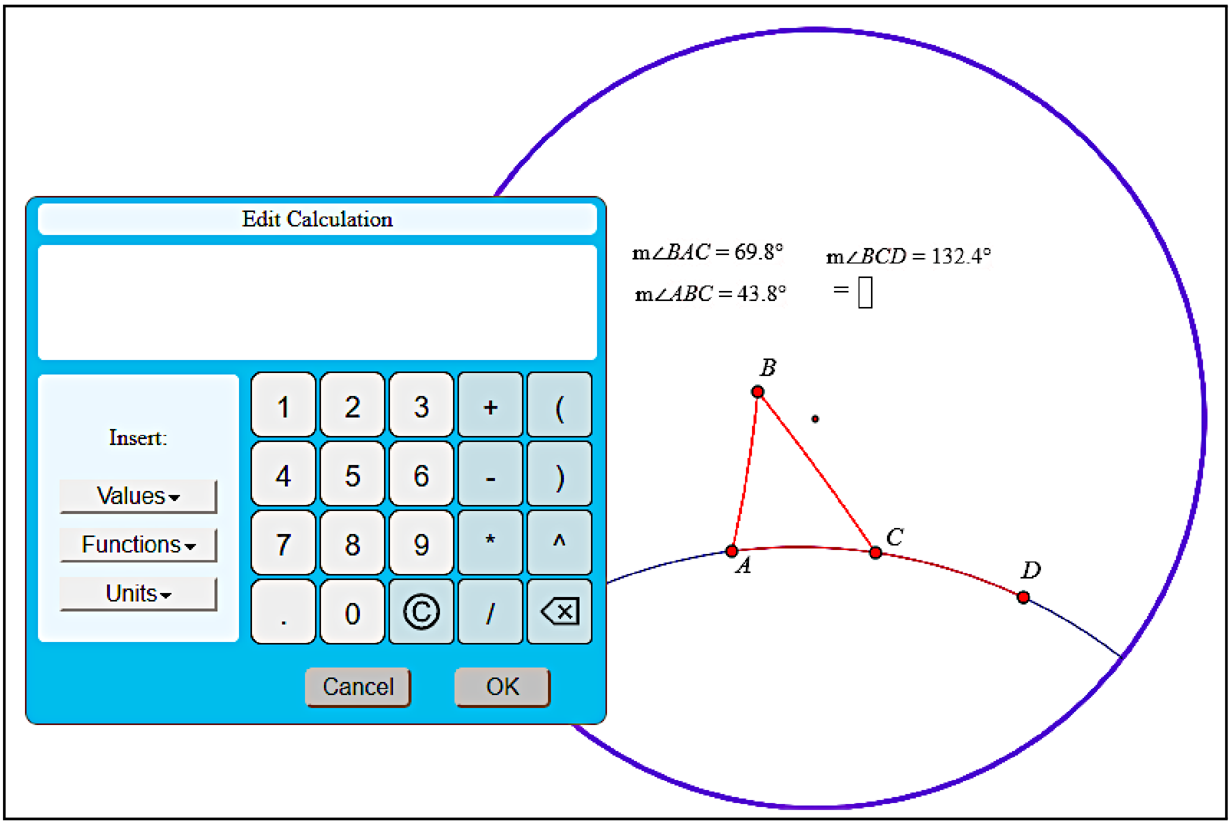


Note: *This construction (created in WSP) demonstrates that in hyperbolic geometry, as in Euclidean geometry, two angles forming a linear pair remain supplementary, and vertical angles remain congruent.*

What about an exterior angle of a triangle? In Euclidean geometry, there are two theorems about exterior angles and their remote interior angles. First, the measure of an exterior angle of a triangle is greater in measure than either of its two remote interior angles. Second, the measure of the exterior angle is equal to the sum of the measures of its two remote interior angles. *Are these two theorems still true in hyperbolic geometry?*

Construct any line and external point. Create the sketch as shown. You will also need the `Measure Angle` tool. To measure an angle, click on the points as if you were naming the angle by three points. For example, to measure, select *A*, *B*, then *C*. The measurements can be moved by clicking and dragging them by the equal sign to any location. To calculate the sum, first select the `Calculate` tool, then the equal sign in the blinking box. This should create a dialog box entitled `Edit Calculation`.

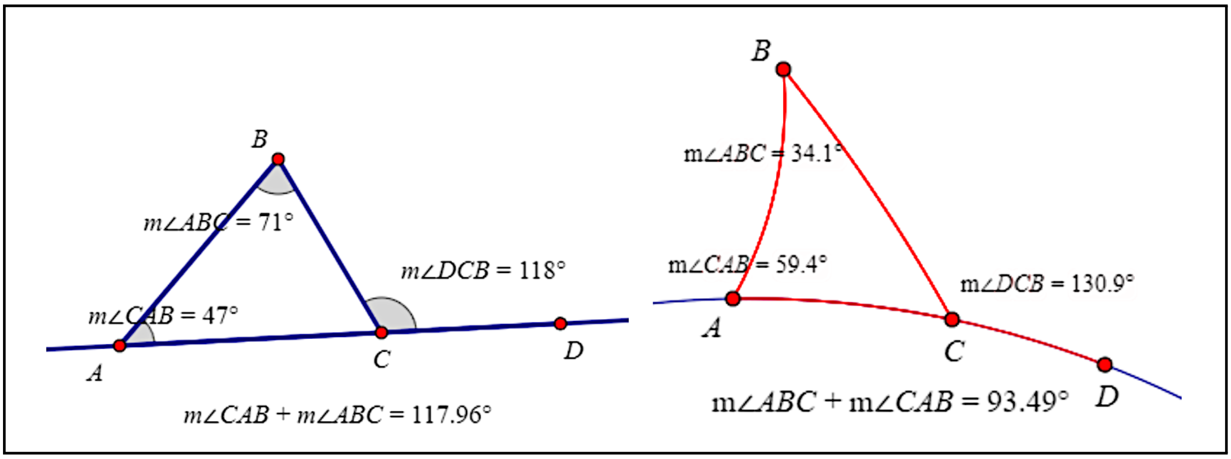
Figure 2
Using the Edit Calculation dialog to sum angles in WSP



To create the sum, click on *m∠BAC*, then +, then *m∠ABC*, then OK. This sum will be dynamic in that, if you drag any of points *A*, *B*, or *C*, the sum will reflect that change in measurements.

Figure 3

The measure of an exterior angle compared to either the measure of a remote interior angle and to the sum of the measures of its two remote interior angles in Euclidean (left) and hyperbolic (right) geometries.

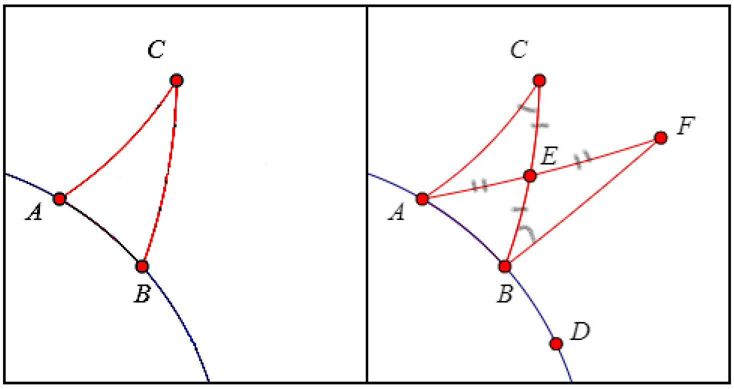


Clearly, the counterexample in hyperbolic geometry is enough to show that the sum of the measures of the remote interior angles is no longer equal to the measure of the exterior angle in hyperbolic geometry.

To show that the measure of an exterior angle is greater than the measure of either remote interior, start with any triangle as shown in Figure 4 (left). Locate a point $D \in \overrightarrow{AB}$ such that $A * B * D$, as shown in Figure 4 (right). Construct the midpoint E of $\overline{BC}$. Locate point $F \in \overline{AF}$ such that $AE = EF$. $\angle AEC$ and $\angle BEF$ form a pair of vertical angles and are therefore congruent. This implies that $\triangle AEC \cong \triangle FEB$ by SAS.

Figure 4

Proof that the measure of an exterior angle is greater than the measure of either remote interior angle.

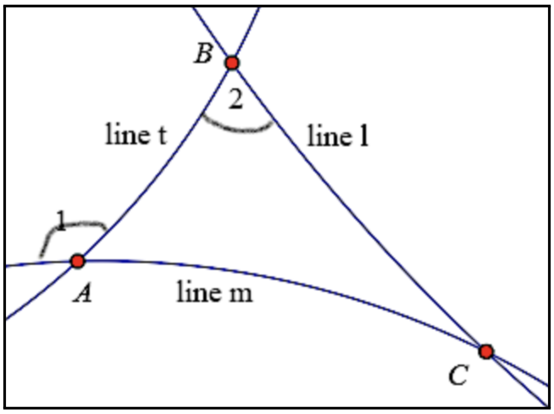


$\overrightarrow{BF}$ is between $\overrightarrow{BC}$ and $\overrightarrow{BD}$ which implies $m\angle CBD = m\angle CBF + m\angle FBD$. Therefore $m\angle CBD > m\angle CBF = m\angle ACB$ (the whole is greater than the part). Repeat the process by extending $\overline{CB}$ to construct the linear-pair partner of $\angle CBD$ and the midpoint of $\overline{AB}$ to prove $m\angle FBD > m\angle BAC$. (This proof also works in Euclidean geometry.)

The last topic of angle relationships for this paper will be the angles formed by two lines and a transversal. It can be proven that if alternate interior angles are congruent, then the two lines are parallel. Suppose that lines l and m are cut by a transversal t such that $\angle 1 \cong \angle 2$, as shown in Figure 5.

Figure 5

Congruent alternate interior angles imply that two lines are parallel.



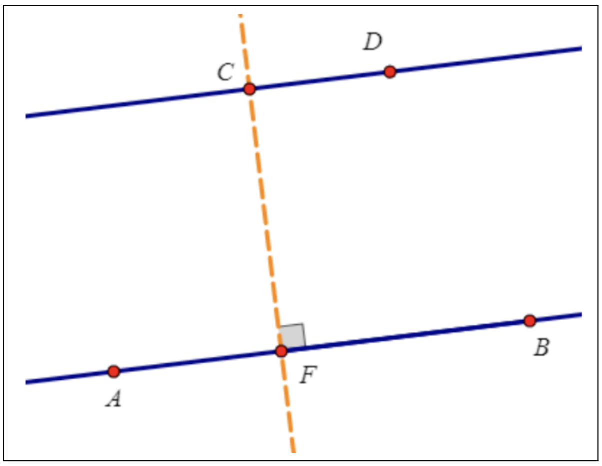
To show that $l \parallel m$, use an indirect proof. Assume that $l \nparallel m$. Then l intersects m somewhere. Without loss of generality, assume that the intersection occurs on the same side as $\angle 2$ at point C. This will construct $\triangle ABC$ where $\angle 1$ is an exterior angle with a remote interior angle of $\angle 2$. This implies $m\angle 1 > m\angle 2$ which is a contradiction since the two angles are congruent. Therefore, it is not the case that $l \nparallel m$, so $l \parallel m$ must be true.

Using vertical angles, linear pairs of angles and transitivity, students can easily show that if the corresponding angles are congruent or that same side interior angles are supplementary, then the lines are parallel in Euclidean geometry. *How do these theorems apply in hyperbolic geometry?*

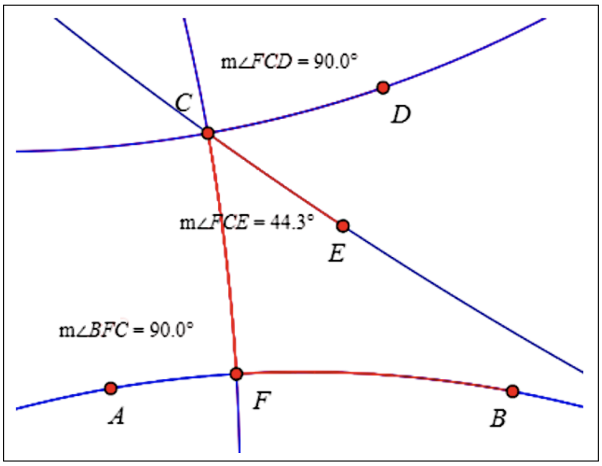
Refer to Figure 6 below. Notice that $\angle AFC$ and $\angle FCD$ would be considered as a pair of alternate interior angles that are also congruent (both are right angles). So $\overline{AB} \parallel \overline{CD}$. Likewise, the other relationships about corresponding angles and same side interior angles would still be true. These three theorems about certain angle relationships implying two lines being parallel are still true in hyperbolic geometry. *But what about their converses?*

Figure 6

Parallel Lines and Alternate Interior Angles



(a) *Euclidean case: two parallel lines cut by a transversal yield congruent alt. interior angles.*



(b) *Hyperbolic case: the alternate interior angles need not be congruent.*

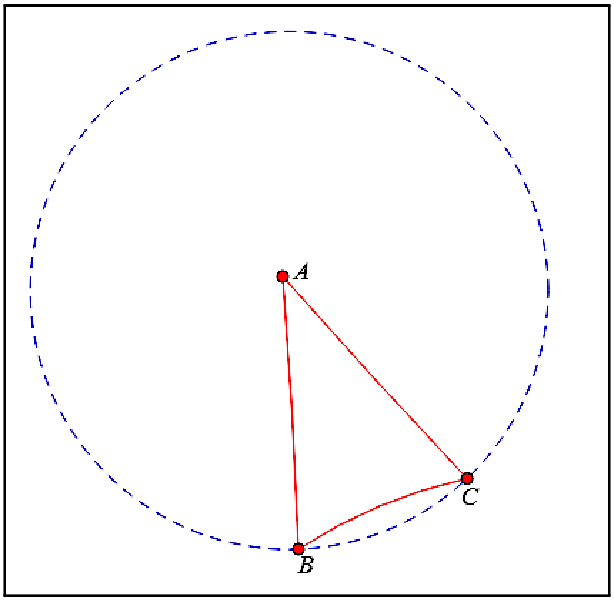
If two parallel lines are cut by a transversal in Euclidean geometry, the alternate interior angles formed are congruent. In Figure 6 (Euclidean), since $\overline{CD} \parallel \overline{AB}$, then $\angle AFC \cong \angle DCF$. In Figure 7 (hyperbolic), since $\overrightarrow{CE}$ is also parallel to $\overline{AB}$, the alternate interior angles formed here would be $\angle AFC$ and $\angle ECF$, which clearly are not equal in measure. As pictured, $m\angle DCF = 90°$ and $\overrightarrow{CE}$ is between rays $\overrightarrow{CD}$ and $\overrightarrow{CF}$, so $\angle ECF$ must be acute and $m\angle ECF < m\angle AFC$. Here the conjecture is disproven by counterexample. This theorem will have a major impact on what is true in Euclidean geometry but not true in hyperbolic geometry.

2 Triangle Properties

There are many properties of triangles that can be investigated. The focus here will be on these theorems: the Isosceles Triangle Theorem, the Scalene Inequality Theorem, the Triangle Inequality Theorem, the angle sum of a triangle, the Pythagorean Theorem, triangle congruence and the points of concurrency.

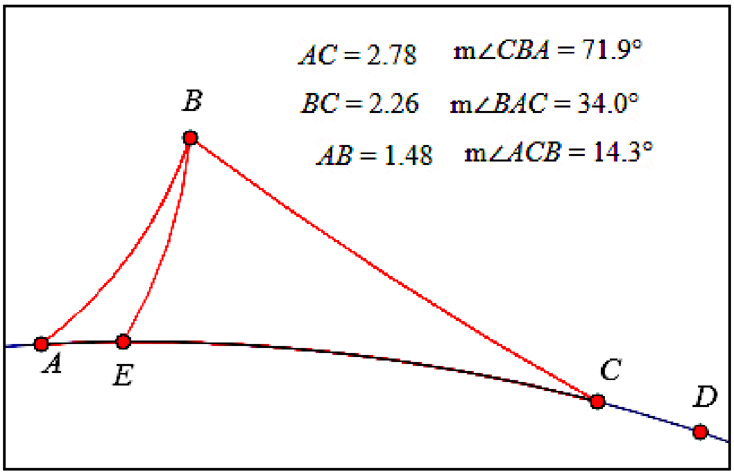
Most students will recall that, in an isosceles triangle, two sides congruent imply the angles opposite are congruent and vice versa. To make this sketch like Figure 7 to demonstrate, use the `Circle and Point` tool in WSP. Make a segment and then use one endpoint as the center of the circle and the other endpoint to set the radius. Choose any point on the circle as the third point of the triangle.

Figure 7
Constructing an isosceles triangle.



The Scalene Inequality Theorem states that the largest side is opposite the largest angle and vice versa. Students can easily construct a triangle to investigate that this theorem remains true. In any $\triangle ABC$, ask the students to measure the three sides of the triangle as in Figure 8. Drag the measurements so that they appear in order from largest to smallest. Then measure the angles and order them in the same manner. *Where are the angles found in relation to the sides? Would it be the same or different if we measured and ordered the angles first?*

Figure 8
Investigating the Scalene Inequality Theorem.

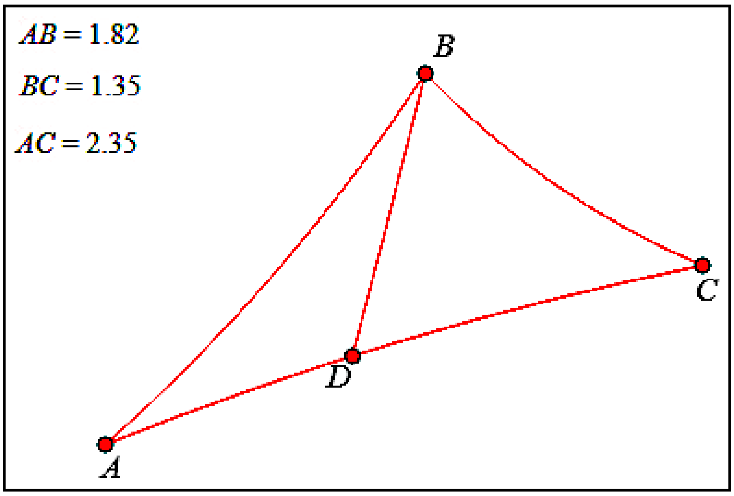


Using Figure 8, since $\triangle ABC$ is scalene, assume $AC > BC$ without loss of generality. Find point $E \in \overline{AC}$ such that $CE = BC$. Now $\triangle BCE$ is isosceles with $\angle EBC \cong \angle CBE$. $m\angle ABC = m\angle ABE + m\angle EBC$ which implies $m\angle ABC > m\angle EBC = m\angle BEC$. But since $\angle BEC$ is an exterior angle of $\triangle ABE$, $m\angle BEC > m\angle A$. Putting the inequalities together, $m\angle ABC > m\angle BEC > m\angle A$.

Now for the other direction, assume $m\angle ABC > m\angle A$. Using Trichotomy, either $AC < BC$, $AC = BC$, or $AC > BC$. From what was just proved, if $AC < BC$ then $m\angle ABC < m\angle A$, which contradicts our assumption. If $AC = BC$, then $m\angle ABC = m\angle A$, which also contradicts our assumption. Therefore, $AC > BC$ must be true.

Likewise, the Triangle Inequality Theorem (the lengths of two sides must sum to be greater than the length of the remaining side) can be tested in a similar sketch (see Figure 9).

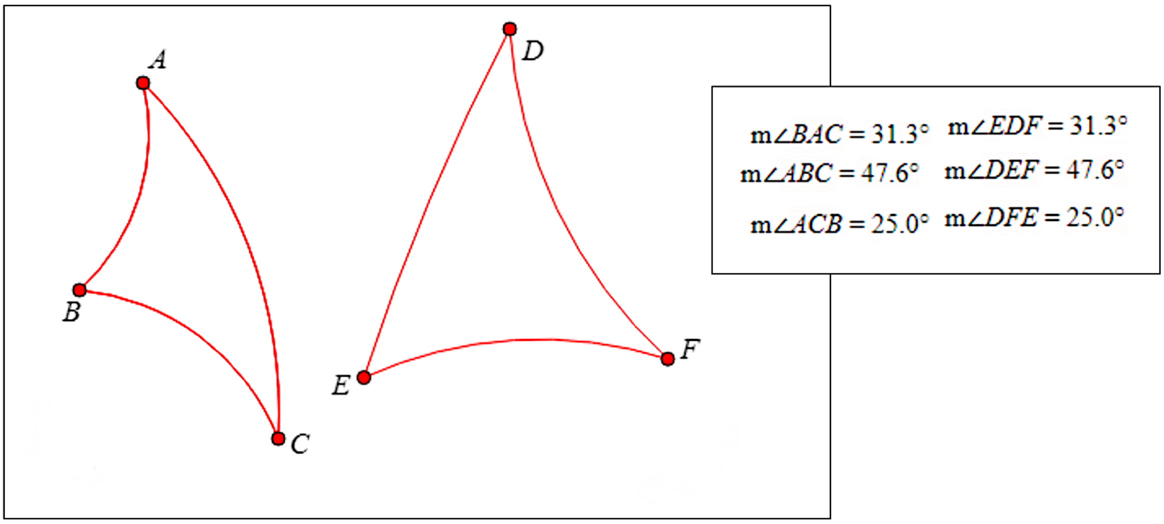
Figure 9
Investigating the Triangle Inequality Theorem.



For the proof, find the point $D \in \overline{AC}$ such that $DC = BC$, creating an isosceles triangle $\triangle BCD$. Using betweenness of rays, $m\angle ABC > m\angle DBC = m\angle BDC$. From the Scalene Inequality Theorem, it follows that $AC > BC$. But $AC = AD + DC > BC$.

Triangle congruency is a very important concept in Euclidean geometry. Many proofs contain triangle congruency in some manner. The valid shortcuts from Euclidean geometry are SSS, SAS, ASA, AAS and HL. Typically, SAS is assumed to be true as a postulate. Students can construct a triangle with the specified congruent corresponding parts to illustrate if the other shortcuts might still be valid. For example, consider the case for SSS. In WSP, construct any hyperbolic triangle . Using the tool `Circle by Center` and `Radius`, each side of $\triangle ABC$ can be duplicated to create $\triangle DEF$. Do so by selecting the tool, then *D*, *A* and *B*. Repeat this with the points *D*, *A* and *C* and also *E*, *B* and *C*. The angles of both triangles can then be measured to see if the corresponding angles are congruent.

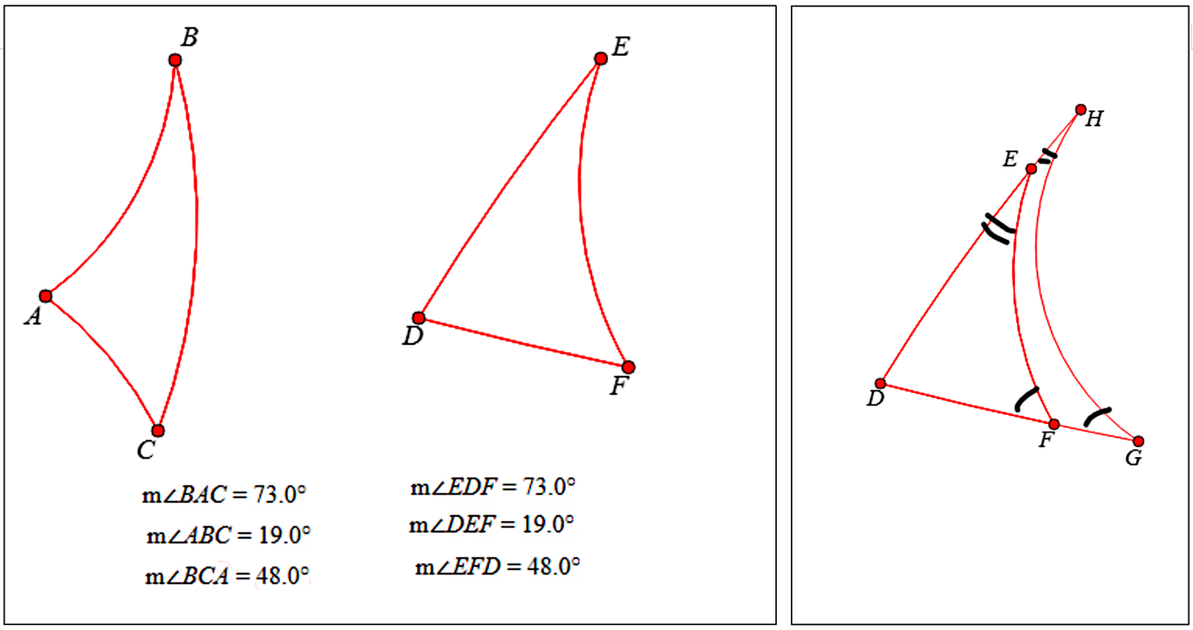
Figure 10
Investigating SSS Congruence.



Logically, a decision can be made about the validity of ASA and AAS. In Euclidean geometry, if one knows the measures of two angles of a triangle, the third angle measure can be calculated by subtracting the other two measures from 180°. That is not the case in hyperbolic geometry. Recall that since the angle sum is strictly less than 180°, the third angle measure cannot be calculated. All three angle measures must be known, which leads us to an additional thought to investigate.

Here is something unexpected though. The figure below shows two triangles in which all the corresponding angles are congruent. Compare the two triangles. *Do they appear to be congruent?*

Figure 11
Investigating AAA Congruence.



Let us assume that the two triangles do have congruent corresponding angles but that the triangles are not congruent. Without loss of generality, assume that $AC > DF$. Find a point G on $\overrightarrow{DF}$ such that $D * F * G$ and $\overline{DG} \cong \overline{AC}$, and a point H on $\overrightarrow{DE}$ such that $D * E * H$ and $\overline{DH} \cong \overline{AB}$ (see the figure to the right above).

Now $\triangle ABC \cong \triangle DHG$ by SAS, which implies that $\angle C \cong \angle G$ and $\angle B \cong \angle H$. But since all of the angles of $\triangle ABC$ are congruent to the corresponding angles of $\triangle DEF$ from our original assumption, $\angle DFE \cong \angle G$ and $\angle DEF \cong \angle H$. We have a linear pair of angles at F, so that means

$$m\angle DFE + m\angle EFG = 180°.$$

By substitution,

$$m\angle G + m\angle EFG = 180°.$$

By the same reasoning,

$$m\angle DEF + m\angle FEH = m\angle H + m\angle FEH = 180°.$$

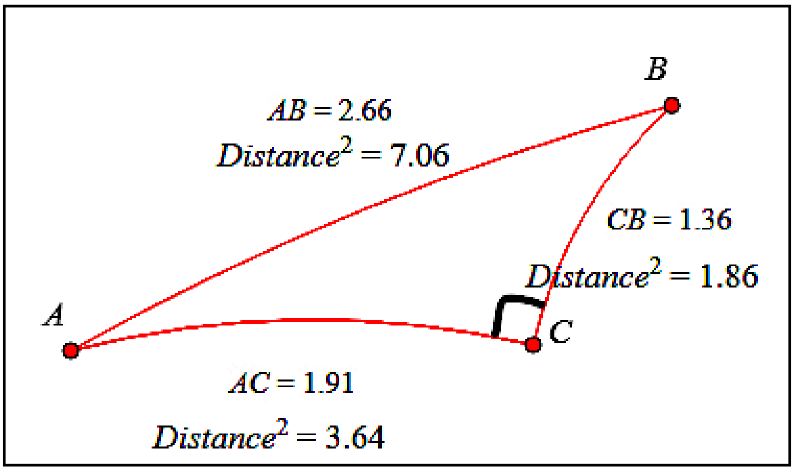
Since $EFGH$ forms a quadrilateral,

$$m\angle G + m\angle EFG + m\angle FEH + m\angle H = 360°.$$

As discussed earlier, the angle sum of a quadrilateral must be strictly less than 360°. So our assumption that the two triangles are not congruent is false, meaning that the two triangles are indeed congruent. Thus, we have proven that AAA is a valid shortcut for triangle congruency.

Since we have now used the fact concerning the angle sum of a quadrilateral, *is the Pythagorean Theorem still valid?* Figure 12 shows a right triangle with the sides measured and lengths squared. It is obvious that the sum of the squares of the two legs will not be equal to the square of the length of the hypotenuse.

Figure 12
Investigating the Pythagorean Theorem.

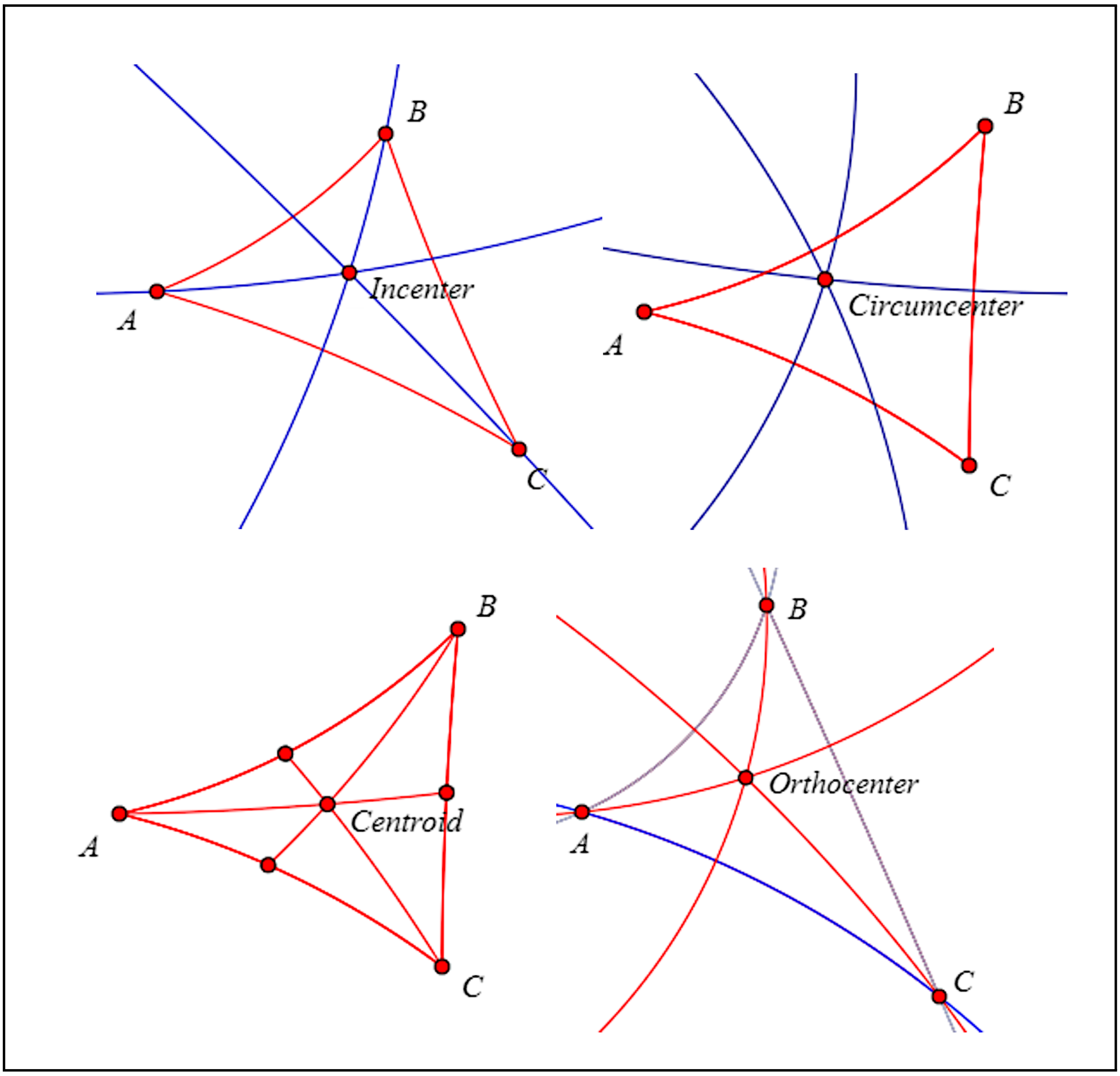


If the traditional Pythagorean Theorem does not hold, *would HL be a valid shortcut for triangle congruency?*

Lastly, we shall consider the points of concurrency of a triangle. The four points of concurrency that are typically studied are the incenter, circumcenter, orthocenter and centroid, found at the intersections of the angle bisectors, perpendicular bisectors, altitudes and medians of a triangle, respectively. As a review of Euclidean geometry, you can easily construct each point on a sketch and investigate any properties, such as equidistance from the sides of the triangle (incenter), equidistance from the vertices (circumcenter), or divides a segment into the same ratio (vertex to centroid: centroid to opposite side = 2:1). Additionally, three of these points are always collinear (circumcenter, orthocenter, and centroid form the Euler Line).

Extending into hyperbolic geometry, you can now construct each of these four points (as shown in Figure 13).

Figure 13
Common Points of Concurrency in Hyperbolic Geometry.

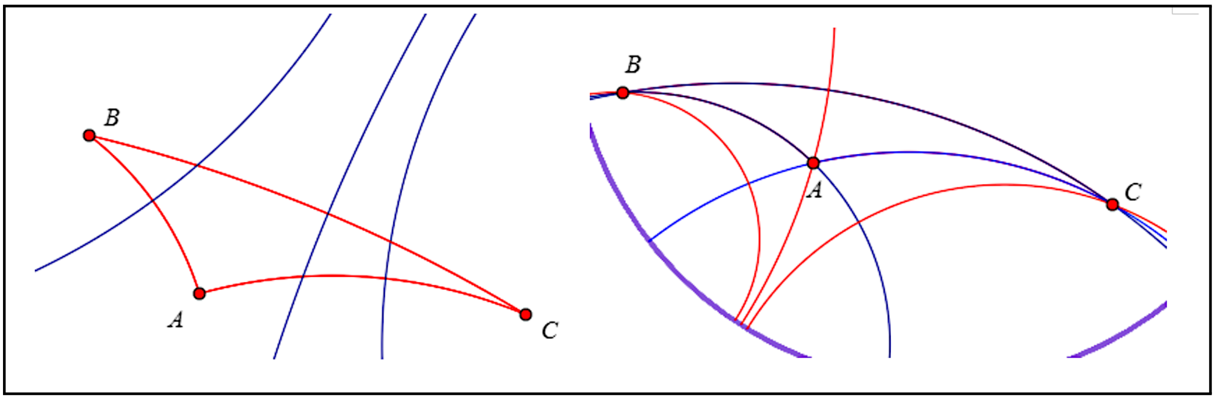


Drag each of the vertices for each point of concurrency. *Do the Euclidean properties still hold in hyperbolic geometry? If not, what has changed? Is the incenter still equidistant from the sides of the triangle? Is the circumcenter still the center of the circumscribed circle?*

Dragging the points for the circumcenter and orthocenter will show that they are not necessarily concurrent. The perpendicular bisectors and altitudes end up being asymptotically parallel, one of two types of parallel lines in hyperbolic geometry. Two lines that are asymptotically parallel are similar to the concept of asymptotes in graphing functions. The distance between the two lines decreases but never becomes zero.

Figure 14

Two Examples Where the Circumcenter (left) and Orthocenter (right) do not exist.



If the circumcenter and the orthocenter do not always exist, can the Euler Line still exist? You should be able to reason to that answer first, then experiment to verify or reject their conjecture.

3 Conclusion

In this second article on hyperbolic geometry, familiar concepts about angle relationships and triangles are investigated. Some concepts remain the same as they were in Euclidean geometry, some have changed or are no longer valid. Students can see that there are more geometries than just Euclidean geometry, that there is a common base to the geometries but each with its distinct differences. The table below shows the similarities and differences of the theorems presented.

Table 1. *Which angle–relations hold in Euclidean vs. Hyperbolic geometry.*

Euclidean Geometry	**Hyperbolic Geometry**
If two angles form a linear pair, the two angles are supplementary.	
If two angles form a pair of vertical angles, the two angles are congruent.	
An exterior angle of a triangle is greater in measure than either of its two remote interior angles.	An exterior angle of a triangle is greater than the sum of the measures of its two remote interior angles.
If two lines are cut by a transversal forming congruent alternate interior angles, then the two lines are parallel.	
If two parallel lines are cut by a transversal, the alternate interior angles are congruent.	If two parallel lines are cut by a transversal, the alternate interior angles are *not* necessarily congruent.
Scalene Inequality Theorem: In any triangle, the largest side is across from the largest angle and vice versa.	
Triangle Inequality Theorem: The sum of the lengths of any two sides is greater than the length of the remaining side.	
The angle sum of a triangle is exactly 180°.	The angle sum of a triangle is strictly less than 180°.
The points of concurrency of a triangle are the incenter, circumcenter, orthocenter, and centroid.	The points of concurrency of a triangle are the incenter and the centroid. The circumcenter and orthocenter do not always exist.

Recommended References

Editor's Note: The items below are suggested for readers who wish to explore further background on dynamic-geometry software and foundational treatments of hyperbolic geometry. Each entry is numbered, and the brief italicized annotation is indented so that it does not "run into" the reference itself.

1. Goldenberg, E. P., & Scher, D. (2014). *WebSketchpad: Dynamic Geometry for the Web*. Key Curriculum. https://www.websketchpad.com

   WebSketchpad is the browser-based dynamic-geometry tool used throughout this article. Readers who wish to reproduce or experiment with the exact compass-and-straightedge constructions in a web environment will find this resource indispensable.

2. Anderson, J. W. (2005). *Hyperbolic Geometry* (2nd ed.). Springer.

   A rigorous, proof-oriented introduction to model-based hyperbolic geometry (e.g. the Poincaré disk). Use this text for a solid foundational understanding of the theorems and constructions used in this article's hyperbolic sketches.

3. Greenberg, M. J. (1993). *Euclidean and Non-Euclidean Geometries: Development and History* (3rd ed.). W. H. Freeman.

   Places Euclidean and hyperbolic geometries in historical perspective. Ideal for readers who want to see how discovery of non-Euclidean axioms led to the constructions described here.

4. Stillwell, J. (1996). *Sources of Hyperbolic Geometry*. American Mathematical Society.

   A collection of the original papers by Beltrami, Poincaré, Klein, etc., that introduced the Poincaré disk and other hyperbolic models. Highly recommended if you wish to see "first-source" proofs.

5. Sinclair, N., & Bruce, C. D. (2015). New opportunities in geometry education at the primary school. *ZDM Mathematics Education*, *47*, 319–329.

   Demonstrates how dynamic-geometry software can be integrated into K–6 curricula. Useful if you plan to introduce WebSketchpad or hyperbolic constructions at the primary level.

Todd O. Moyer is Professor of Mathematics at Towson University. Dr. Moyer's primary research interests center on the use of technology in geometry instruction, particularly dynamic-geometry software (e.g., WebSketchpad, GeoGebra) to improve students' conceptual understanding and procedural fluency. Dr. Moyer was an early adopter of WebSketchpad in secondary classrooms. He also serves on the editorial board of the *Ohio Journal of School Mathematics* and is a member of the Maryland Council of Teachers of Mathematics.