Representing Odd Integers as Deficient Rectangles

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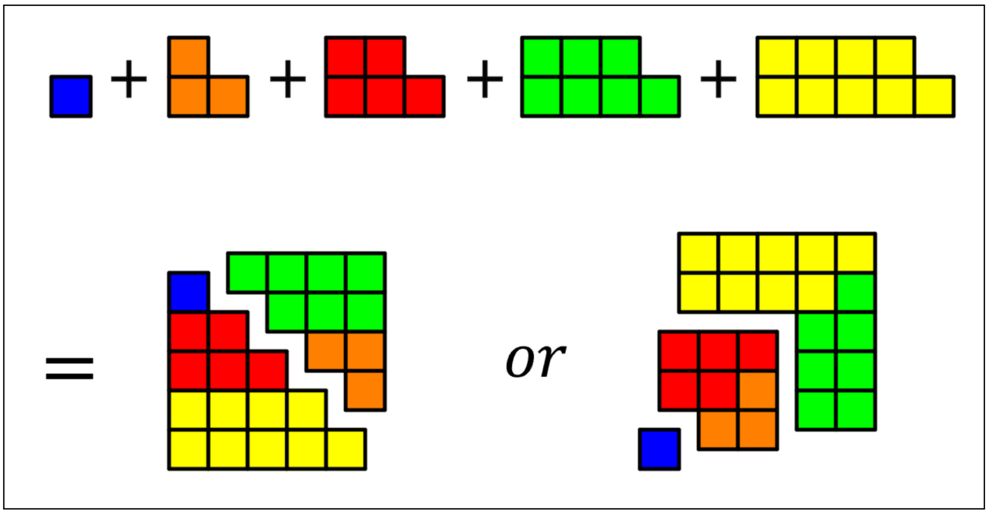
Abstract

By representing odd integers as deficient $2 \times k$ rectangles, the author illustrates that the sum of the first n odd integers forms a square.

Keywords: odd integers, deficient rectangles, sum of integers, geometric representation

Figure 1: Visual proof for odd case: sum of first n odd integers equals n^2.

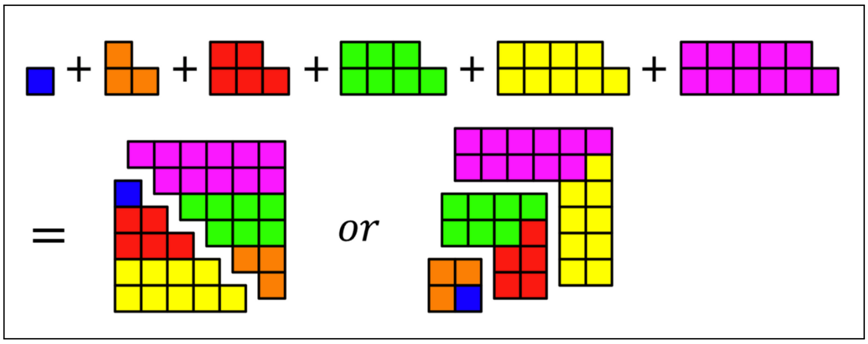


Note: By representing odd integers as deficient $2 \times k$ rectangles, we can visually demonstrate that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ when n is odd.

Figures 1 and 2 demonstrate the visual proof using deficient rectangles for odd and even values of n, respectively. This geometric approach is one of several methods for proving the identity $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. Nelsen's collection of Proofs Without Words (Nelsen, 1993; Nelsen, 2015) presents multiple visual demonstrations of this identity using various geometric interpretations, including L-shaped pieces, diagonal arrangements and vertical rectangles. Additional proofs using different geometric approaches can be found in Moreno (2017) and Sangwin and Tanswell (2023).

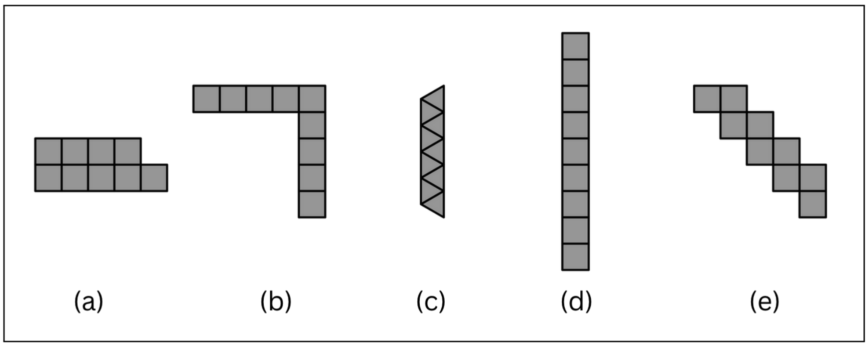
Figure 3 illustrates five different ways to represent the odd integers geometrically. The visual proof depicted in Figures 1 and 2 is based on the representation labeled (a). Nelsen (1993) provides alternative proofs for the same identity based on the representations labeled (b), (c) and (d). Nelsen (2015) provides another proof based on configuration (c) and another one based on configuration (d). Moreno (2017) provides yet another proof based on (d). Sangwin and Tanswell (2023) provide a proof based on (e).

Figure 2: Visual proof for even case: sum of first n odd integers equals n^2.



Note: The same principle applies when n is even, showing how deficient rectangles can be arranged to form perfect squares.

Figure 3: Different geometric representations of odd integers.



Note: Five different ways to represent odd integers: (a) deficient rectangles, (b) L-shaped pieces, (c) diagonal arrangements, (d) vertical rectangles, and (e) staircase patterns.

The visual demonstration shows that:

$$1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

References

Nelsen, R. B. (1993). *Proofs Without Words: Exercises in Visual Thinking*. The Mathematical Association of America.

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Moreno, S. G. (2017). Proofs Without Words: Sums of Odd Integers. *Mathematics Magazine*, *90*(4), 298.

Sangwin, C., & Tanswell, F. S. (2023). Developing new picture proofs that the sums of the first n odd integers are squares. *The Mathematical Gazette*, *107*(569), 249–262.

Stephan Berendonk is Professor of Mathematics Education at the University of Wuppertal. His research highlights the aesthetic and creative dimensions of mathematical reasoning. He develops visual proofs for elementary results and designs learning environments that bring out essential features of mathematical thinking.