An Elementary Difference–Matrix Evaluation of the Series $\sum_{n=1}^{\infty} \frac{n^k}{2^n}$

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Abstract

We present a fully elementary method for evaluating the infinite series $S_k = \sum_{n=1}^{\infty} \frac{n^k}{2^n}$, where k is a fixed natural number. The method relies only on repeated scaling, term-by-term subtraction, and the systematic use of finite differences. No tools from calculus, generating functions, or special functions are required. Starting from explicit computations for $k = 1, 2, 3, 4$, we show how a stable pattern emerges and how this pattern can be described and proved using a difference matrix. Finally, we present an interesting combinatorial identity.

Keywords: Series, Difference Matrix, Combinatorial Identity

1 The Starting Point: A Simple but Powerful Idea

We begin with the series

$$S_k = \frac{1^k}{2} + \frac{2^k}{4} + \frac{3^k}{8} + \frac{4^k}{16} + \cdots$$

The key idea is extremely simple: Multiply the entire series by $\frac{1}{2}$, do not simplify any of the fractions, then subtract the result from the original series term by term. This idea already works perfectly for $k = 1$.

1.1 Example

In case $k = 1$ we have:

$$S_1 = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$$

Multiplying by $\frac{1}{2}$:

$$\frac{1}{2}S_1 = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \cdots$$

Now subtract column by column:

$$S_1 - \frac{1}{2}S_1 = \left(\frac{1}{2} - 0\right) + \left(\frac{2}{4} - \frac{1}{4}\right) + \left(\frac{3}{8} - \frac{2}{8}\right) + \cdots$$

This gives

$$\frac{1}{2}S_1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

and therefore $S_1 = 2$.

1.2 Why Higher Powers Are Not Automatic

For $k \geq 2$, the same idea works, but it must be carried out carefully and repeatedly. One cannot be expected to guess which fractions to subtract; therefore, we explicitly write out every step.

We now explain this process in detail for $k = 2, 3, 4$. These examples are essential for understanding the general method.

1.2.1 The Case $k = 2$: Writing Everything Explicitly

$$S_2 = \frac{1}{2} + \frac{4}{4} + \frac{9}{8} + \frac{16}{16} + \frac{25}{32} + \cdots$$

Multiplying by $\frac{1}{2}$:

$$\frac{1}{2}S_2 = \frac{1}{4} + \frac{4}{8} + \frac{9}{16} + \frac{16}{32} + \cdots$$

Subtract term by term:

$$S_2 - \frac{1}{2}S_2 = \left(\frac{1}{2} - 0\right) + \left(\frac{4}{4} - \frac{1}{4}\right) + \left(\frac{9}{8} - \frac{4}{8}\right) + \cdots$$

This produces

$$\frac{1}{2}S_2 = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + \frac{7}{16} + \cdots$$

Repeating the process once more:

$$\frac{1}{4}S_2 = \frac{1}{4} + \frac{3}{8} + \frac{5}{16} + \frac{7}{32} + \cdots$$

Subtracting again:

$$\frac{1}{2}S_2 - \frac{1}{4}S_2 = \left(\frac{1}{2} - 0\right) + \left(\frac{3}{4} - \frac{1}{4}\right) + \left(\frac{5}{8} - \frac{3}{8}\right) + \cdots$$

Thus

$$\frac{1}{4}S_2 = \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{2} + 1$$

and therefore $S_2 = 6$.

1.2.2 The Case $k = 3$

Here three subtractions are needed.

$$S_3 = \frac{1}{2} + \frac{8}{4} + \frac{27}{8} + \frac{64}{16} + \cdots$$

We now form the four series

$$S_3, \quad \frac{1}{2}S_3, \quad \frac{1}{4}S_3, \quad \frac{1}{8}S_3$$

and subtract each one from the previous one, always term by term. After the third subtraction, we obtain

$$\frac{1}{8}S_3 = \frac{1}{2} + \frac{5}{4} + \frac{6}{8} + \frac{6}{16} + \frac{6}{32} + \cdots$$

From this point onward, all numerators are equal to 6, while the denominators continue to double. The infinite tail is therefore a geometric series. Carrying out the arithmetic gives

$$S_3 = 26.$$

1.2.3 The Case $k = 4$: Seeing the Pattern Clearly

$$S_4 = \frac{1}{2} + \frac{16}{4} + \frac{81}{8} + \frac{256}{16} + \cdots$$

Repeating the subtraction process four times leads to

$$\frac{1}{16}S_4 = \frac{1}{2} + \frac{12}{4} + \frac{23}{8} + \frac{24}{16} + \frac{24}{32} + \cdots$$

The numerators form the sequence

$$1, 12, 23, 24, 24, 24, \ldots$$

From this point onward, the numerators are constant and the denominators double. As you can see, the sequences of numbers that appear in the numerators of $\frac{1}{2^k}S_k - \frac{1}{2^{k+1}}S_k$ exhibit a similar behavior. Now we focus on the sequence of numbers that appear in the numerators.

2 The Difference Matrix and Its Powers

To explain the origin of these numerators, we introduce the difference matrix Δ. Acting on an infinite column vector $(a_1, a_2, a_3, \ldots)^T$, the matrix Δ is given by

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus, multiplication by Δ produces the first forward differences.

Powers of the Difference Matrix

By direct matrix multiplication one obtains

$$\Delta^2 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & \cdots \\ 0 & 1 & -2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\Delta^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -3 & 1 & 0 & 0 & \cdots \\ 3 & -3 & 1 & 0 & \cdots \\ -1 & 3 & -3 & 1 & \cdots \\ 0 & 1 & -3 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$\Delta^4 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -4 & 1 & 0 & 0 & \cdots \\ 6 & -4 & 1 & 0 & \cdots \\ -4 & 6 & -4 & 1 & \cdots \\ 1 & -4 & 6 & -4 & \cdots \\ 0 & 1 & -4 & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In the general case, we have $\Delta^k =$

$$\begin{pmatrix} \binom{k}{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\binom{k}{1} & \binom{k}{0} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \binom{k}{2} & -\binom{k}{1} & \binom{k}{0} & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\binom{k}{3} & \binom{k}{2} & -\binom{k}{1} & \binom{k}{0} & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ (-1)^k\binom{k}{k} & (-1)^{k-1}\binom{k}{k-1} & \vdots & \vdots & -\binom{k}{1} & \binom{k}{0} & 0 & 0 & \cdots \\ 0 & (-1)^k\binom{k}{k} & (-1)^{k-1}\binom{k}{k-1} & \vdots & \vdots & -\binom{k}{1} & \binom{k}{0} & 0 & \cdots \\ 0 & 0 & (-1)^k\binom{k}{k} & (-1)^{k-1}\binom{k}{k-1} & \vdots & \vdots & -\binom{k}{1} & \binom{k}{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Note that the numbers appearing here evoke Pascal's triangle.

2.1 Action on Power Sequences

Applying these matrices to the column vector $(n^k) = (1^k, 2^k, 3^k, \dots)^T$ yields exactly the finite-difference sequences. For example,

$$\Delta^4(n^4) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -4 & 1 & 0 & 0 & \cdots \\ 6 & -4 & 1 & 0 & \cdots \\ -4 & 6 & -4 & 1 & \cdots \\ 1 & -4 & 6 & -4 & \cdots \\ 0 & 1 & -4 & 6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ 16 \\ 81 \\ 256 \\ 625 \\ 1296 \\ \vdots \end{pmatrix} = 1, 12, 23, 24, 24, 24, \dots$$

In fact we have:

$$\begin{aligned} (n^4) &: 1, 16, 81, 256, 625, \dots \\ \Delta(n^4) &: 1, 15, 65, 175, 369, \dots \\ \Delta^2(n^4) &: 1, 14, 50, 110, 194, \dots \\ \Delta^3(n^4) &: 1, 13, 36, 60, 84, \dots \\ \Delta^4(n^4) &: 1, 12, 23, 24, 24, 24, \dots \end{aligned}$$

The sequence becomes constant and equal to $4! = 4 \times 3 \times 2 \times 1$. This phenomenon holds in general: for every k, the sequence $\Delta^k(n^k)$ stabilizes at the value $k!$.

3 Derivation of the General Formula

We now explain how the general formula arises. Each subtraction step corresponds to multiplying the series by $\frac{1}{2}$ and subtracting term by term. After performing this operation exactly k times, every term of the resulting series has the form

$$\frac{(\Delta^k(n^k))_n}{2^n},$$

where $(\Delta^k(n^k))_n$ is the nth term of the sequence $\Delta^k(n^k)$. Thus we obtain

$$\frac{1}{2^k}S_k = \sum_{n=1}^{\infty} \frac{(\Delta^k(n^k))_n}{2^n}.$$

From the theory of finite differences, we know that

$$(\Delta^k(n^k))_n = k! \quad \text{for all } n \geq k+1.$$

Therefore, the series splits naturally into two parts:

$$\frac{1}{2^k}S_k = \sum_{n=1}^{k} \frac{(\Delta^k(n^k))_n}{2^n} + \sum_{n=k+1}^{\infty} \frac{k!}{2^n}.$$

The second sum is a geometric series and can be evaluated explicitly:

$$\sum_{n=k+1}^{\infty} \frac{k!}{2^n} = \frac{k!}{2^k}.$$

Multiplying both sides by 2^k, we finally obtain

$$S_k = 2^k \sum_{n=1}^{k} \frac{(\Delta^k(n^k))_n}{2^n} + k! \tag{1}$$

Every term in this formula has been explicitly constructed, and no hidden steps remain.

3.1 Worked Examples

3.1.1 Example 1: $k = 2$

$$\Delta^2(n^2) = 1, 2, 2, 2, \dots$$

$$S_2 = 4\left(\frac{1}{2} + \frac{2}{4}\right) + 2 = 6.$$

3.1.2 Example 2: $k = 3$

$$\Delta^3(n^3) = 1, 5, 6, 6, 6, \dots$$

$$S_3 = 8\left(\frac{1}{2} + \frac{5}{4} + \frac{6}{8}\right) + 6 = 26.$$

3.1.3 Example 3: $k = 4$

$$\Delta^4(n^4) = 1, 12, 23, 24, 24, \dots$$

$$S_4 = 16\left(\frac{1}{2} + \frac{12}{4} + \frac{23}{8} + \frac{24}{16}\right) + 24 = 150.$$

4 An Interesting Combinatorial Identity

Since the sequence of numbers in the numerators becomes stable and constant beyond a certain point, we can derive a combinatorial identity of the following form. $(\Delta^k(n^k))^T =$

$$\left(\begin{array}{l}
\binom{k}{0}1^k \\
\binom{k}{0}2^k - \binom{k}{1}1^k \\
\binom{k}{0}3^k - \binom{k}{1}2^k + \binom{k}{2}1^k \\
\binom{k}{0}4^k - \binom{k}{1}3^k - \binom{k}{2}2^k + \binom{k}{3}1^k \\
\vdots \\
\binom{k}{0}k^k - \binom{k}{1}(k-1)^k + \binom{k}{2}(k-2)^k - \binom{k}{3}(k-3)^k + \cdots + (-1)^{k-1}\binom{k}{k-1}(k-(k-1))^k \\
\binom{k}{0}(k+1)^k - \binom{k}{1}(k)^k + \binom{k}{2}(k-1)^k - \binom{k}{3}(k-2)^k + \cdots + (-1)^{k-1}\binom{k}{k-1}2^k + (-1)^k\binom{k}{k}1^k \\
\binom{k}{0}(k+2)^k - \binom{k}{1}(k+1)^k + \binom{k}{2}(k)^k - \binom{k}{3}(k-1)^k + \cdots + (-1)^{k-1}\binom{k}{k-1}3^k + (-1)^k\binom{k}{k}2^k \\
\vdots \\
\binom{k}{0}(k+m)^k - \binom{k}{1}(k+m-1)^k + \binom{k}{2}(k+m-2)^k + \cdots + (-1)^{k-1}\binom{k}{k-1}(m+1)^k \\
\vdots
\end{array}\right)$$

Now, since from the kth row onward all the values are constant and equal to $k!$, we have:

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k+m-i)^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^k = k! \tag{2}$$

4.1 Example

As a concrete illustration, consider the cases $k = 3, 5$. A direct computation shows that

$$\sum_{i=0}^{3}(-1)^i\binom{3}{i}(9-i)^3 = \sum_{i=0}^{3}(-1)^i\binom{3}{i}(6-i)^3 = 3!.$$

Similarly, we obtain

$$\sum_{i=0}^{5}(-1)^i\binom{5}{i}(15-i)^5 = \sum_{i=0}^{5}(-1)^i\binom{5}{i}(10-i)^5 = 5!.$$

These calculations demonstrate that the value of the sum does not depend on the parameter m.

5 Conclusion

By insisting on explicit term-by-term subtraction and by carefully tracking numerators and denominators, we obtain a complete and elementary evaluation of the series $\sum n^k/2^n$. The difference-matrix viewpoint explains why the computation always terminates after k steps and why factorials naturally appear. The method provides a concrete bridge between arithmetic, finite differences, and infinite series.

Mahdi Imaninezhad is a mathematics teacher and educator in Gonabad, Iran, with extensive experience in secondary schools and universities. He completed the PhD comprehensive examination in Mathematics (Functional Analysis) and holds an M.Sc. in Mathematics (Analysis) and a B.Sc. in Pure Mathematics. His research interests focus on pure mathematics and on making advanced concepts accessible to students.