The Simplex Method for Systems of Linear Inequalities

Todd O. Moyer, *Towson University*

Abstract: *This article details the application of the Simplex Method for an Algebra 2 class. Students typically learn how to use the `rref` button on the graphing calculator. This leads the students to believe that every matrix can be row reduced in that manner. The Simplex Method is an example of not using the `rref` button, instead requiring students to recall the elementary row operations to be performed on matrices. Instructions on using the TI-84 calculator commands are given at the end of the article.*

Keywords. *Matrix algebra, systems of equations, graphing calculator*

1 Introduction

A high school curriculum in Algebra 2 contains both solving systems of linear equations, reducing matrices, and then systems of linear inequalities. Students learn linear combination techniques and then elementary row operations for matrices. Teachers will typically share how to use a command, such as `rref` on the TI-83/84 family of graphing calculators, to go from the original matrix directly to the solution matrix. The systems of linear inequalities usually contain two variables and are solved graphically. Students graph the inequalities, calculate the points of intersection, and then substitute those coordinates into the equation to be optimized. These systems of inequalities rarely involve three independent variables, so students may see no other method for problems of this sort. Thus, students may think that every matrix can be reduced by pressing only one button, `rref`. One topic in mathematics that is somewhat advanced but accessible to most high school students is the Simplex Method. It allows for matrices to be used in systems of linear inequalities and reinforces the skills of using elementary row operations.

Both the National Council of Teachers of Mathematics (NCTM) and the Common Core State Standards for Mathematics (CCSSM) recommend that students study these concepts at some point in their high school career. (Please refer to HSA.REI.D.12, HSA.REI.C.8, and HSA.REI.C.6 and NCTM, p. 296). This article also addresses the Standards for Mathematical Practice 1, 4 and 5.

1.1 Teaching observations

During my high school teaching career, I made a few observations of students studying these topics. Most students understand the “how to” in reducing matrices into reduced row-echelon form. Students typically make mistakes in performing the arithmetic, not the algebra. Using a graphing calculator can utilize the conceptual understanding of reducing matrices without points lost from arithmetic errors. For the purposes of this article, it is assumed that students have already learned how to solve matrices through the elementary row operations and row reductions through linear combination. It is best if students have learned how to create a leading 1 in a column and use that particular row to make all other column entries a 0. Since students make arithmetic errors frequently when row reducing by hand, I suggest that the calculator commands of *row, row+, and *row+ (explained and demonstrated in the appendix) to eliminate the arithmetic mistakes. In other words, the calculator becomes a "convenience box," where a student still needs to know what to do (conceptual understanding) but the calculator acts like a black box in performing the calculations.

1.2 The Simplex Method

The Simplex Method describes a method for entering inequalities as equations into a matrix and the keys to row-reduce that matrix. This method also is applicable to any number of independent variables, making graphing unnecessary. The following is a demonstration of an activity, including calculator commands, which make arithmetic errors nonexistent.

2 The USB Manufacturing Problem

Consider the following problem.

> A very small department of a manufacturing company makes two products, plastic cases for USB drives with two ports and cases with one port. Each two-port case takes 2 hours to assemble, whereas a one port case requires 1 hour. There are a maximum of 50 hours of labor dedicated for the day. The cost of materials for two-port case is $1; the material for a one-port case costs $2. Production only allows a maximum of $70 per day. Find the number of two-port cases and one-port case produced that will maximize the profit. Each two-case port earns a profit of $4 and each one-port case $5.

When I taught students the Simplex Method, I did it side by side with a problem that had been already solved by hand. As such, students would write each inequality, then graph the inequalities and locate the intersection points as shown below. [Note: I suggest that students shade in such a manner that the feasible region is clearly identified. That requires graphing the inequalities in reverse order as evidenced below. This enables a student to trace and find possible solutions much easier than when the feasible region is the area shaded by everything. The feasible region is the region in white.] If desired, a student may plot a point found within the feasible region, such as $(10, 10)$, to determine its profit ($90) and help verify that a point from the feasible region is a viable solution.

The four points of intersection or corner points are $(0, 0)$, $(25, 0)$, $(10, 30)$ and $(0, 35)$. A student would now substitute each of those four points into the optimization equation to find the maximum profit. Respectively, the profits would be $0, $100, $190, and $170. So the optimal solution would be 10 two-port cases and 30 one-port cases a day for a daily profit of $190. Let x = the number of two-port cases produced and y = the number of one-port cases produced. Then we see that we need to maximize $M = 4x + 5y$ subject to the following constraints:

$$\begin{aligned} 2x + y &\leq 50 \\ x + 2y &\leq 70 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

A graph of this system of constraints is illustrated in Figure 1.

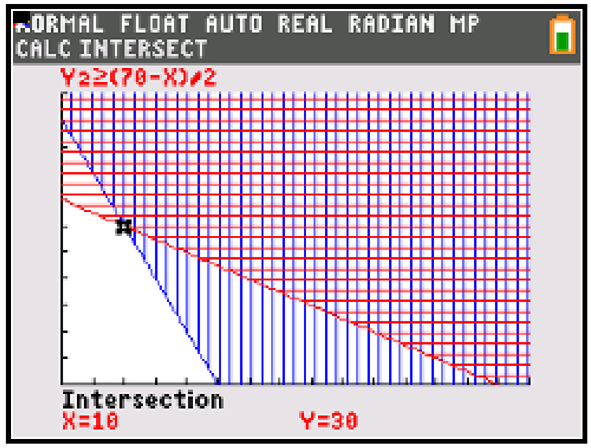


Fig. 1: *Graph of system from TI-84 calculator.*

3 Introducing the Simplex Method

Now we shall demonstrate the Simplex Method. I use the Simplex Method to demonstrate a method for finding an answer in a more direct manner than graphing and solving to find the intersections, and evaluating the optimization function at each corner point. The proof of the method is beyond the scope of Algebra 2. I will explain the steps of what to do.

In order to place inequalities into a matrix, each inequality must become an equation. To do this, a slack variable is introduced to pick up the "slack" for equality. Each inequality becomes as follows:

$$2x + y \leq 50 \rightarrow 2x + y + s_1 = 50$$
$$x + 2y \leq 70 \rightarrow x + 2y + s_2 = 70$$

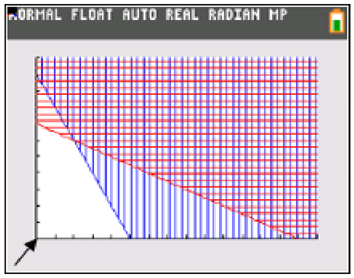
For example, if $x = 10$ and $y = 10$, then $s_1 = 20$, the amount needed for the first inequality to achieve equality. In a practical interpretation, there are 20 hours of labor remaining. For the second inequality, $s_2 = 40$, or \$40 leftover. This shows that perhaps all of the resources are being used to their fullest extent.

We can now begin to create the matrix. Each of the above inequalities can be placed into the matrix as a row, with the inclusion of the slack variables making each row an equation. The third row is the optimization equation, $M = 4x + 5y \rightarrow -4x + -5y + M = 0$, solved so that all variables are on one side, with having a positive coefficient. This matrix, illustrated in Figure 2, is called the Simplex Tableau. The columns are x, y, s_1, s_2, M and the constants, respectively.

$$\begin{array}{cccccc} x & y & s_1 & s_2 & M & c \end{array}$$
$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 50 \\ 1 & 2 & 0 & 1 & 0 & 70 \\ -4 & -5 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Fig. 2: *Simplex Tableau.*

If a column contains a 1 and all other entries are 0, then that variable is considered to be defined. At this point, $s_1 = 50$, $s_2 = 70$ and $M = 0$. All other variables are considered to be 0. Currently, the company would produce 0 two-port cases and 0 one-port case for a profit of \$0. Notice, as Figure 3 suggests, that we are now located at the origin of the graph, one of the original corner points.



$$\begin{array}{cccccc} x & y & s_1 & s_2 & M & c \end{array}$$
$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 50 \\ 1 & 2 & 0 & 1 & 0 & 70 \\ -4 & -5 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$x=0$; $y=0$; $s_1 = 50$, $s_2 = 70$; $M = 0$

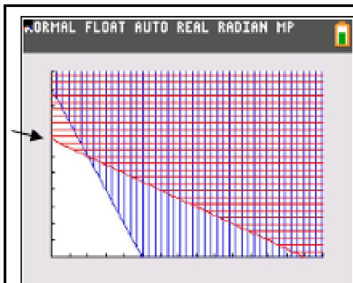
Fig. 3: *Problem scenario with $x = 0$, $y = 0$, $s_1 = 50$, $s_2 = 70$ and $M = 0$*

A pivot needs to be established within the matrix. The pivot is the entry about which we will create a 1 and all other entries become 0. First, identify the column with the most negative entry in the last row. (If there is a tie, choose either one.) Once a column is identified, choose the least nonnegative ratio of the constant term divided by the row entries in that column. This approach is suggested in Figure 4.

$$\begin{array}{cccccc} x & y & s_1 & s_2 & M & c \end{array}$$
$$\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 50 \\ 1 & 2 & 0 & 1 & 0 & 70 \\ -4 & -5 & 0 & 0 & 1 & 0 \end{bmatrix} \qquad \frac{50}{1} = 50 \quad \frac{70}{2} = 35$$

Fig. 4: *Establishing a pivot.*

Our pivot will be around the 2, 2 entry. That entry needs to become a 1. Using elementary row operations, we would first multiply the second row by $\frac{1}{2}$. Then multiply row 2 by -1 and add it to row 1, and likewise multiply row 2 by 5 and add it to row 3, resulting in the following matrix and solutions of 0 two-port cases and 35 one-port cases yielding a profit of \$175. Refer to Figure 5. Notice how we have moved to one of the adjacent original corner points.



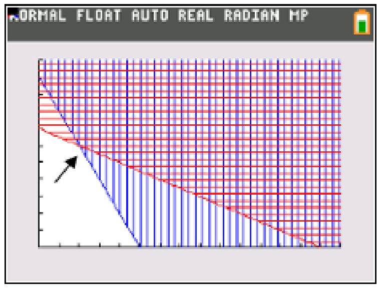
$$\begin{array}{cccccc} x & y & s_1 & s_2 & M & c \end{array}$$
$$\begin{bmatrix} 3/2 & 0 & 1 & -1/2 & 0 & 15 \\ 1/2 & 1 & 0 & 1/2 & 0 & 35 \\ -3/2 & 0 & 0 & 5/2 & 1 & 175 \end{bmatrix}$$

$x = 0$; $y = 35$; $s_1 = 15$; $s_2 = 0$; $M = 175$

Fig. 5: *Moving to an adjacent corner point.*

The first column contains a negative entry in the last row. Check the ratios again between constant and entry in the first column. In this case, the ratios would be $\frac{15}{1.5}$ and $\frac{35}{0.5}$. Our next pivot will be the

1, 1 entry because the first column contained the most negative number in the last row and the first row had the least ratio. This is illustrated in Figure 6.



$$\begin{array}{cccccc} x & y & s_1 & s_2 & M & c \end{array}$$
$$\begin{bmatrix} 1 & 0 & 2/3 & -1/3 & 0 & 10 \\ 0 & 1 & -1/3 & 2/3 & 0 & 30 \\ 0 & 0 & 1 & 2 & 1 & 190 \end{bmatrix}$$

$x = 10$; $y=30$; $s_1 = 0$; $s_2 = 0$; $M = 190$

Fig. 6: *Moving to the next pivot.*

The Simplex Method continues by first identifying the column with the most negative entry in the last row, then determining the pivot through the ratio process until no negative entries remain in the last row. Since there are no more negative entries in the last row, the final solution is that 10 two-port cases and 30 one-port cases yield a maximum profit of $190. Since both slack variables are undefined, there are no labor hours or financing unused. Notice how the process leads us around the corner points until the optimum solution is found.

4 In Summary

What was demonstrated here is a basic problem using the Simplex Method. Not treated here were equal negative values in the last row (pick either one) or minimization problems (work with the dual). The point of this activity is to practice the skills of reducing a matrix by creating leading 1s within a particular column. It also shows that not every matrix can be solved by using the `rref` command on the graphing calculator. This activity would be appropriate for an Algebra 2 class or perhaps an advanced Algebra 1. The Simplex Method is an important topic to be covered. First, it demonstrates a more efficient method of solving linear programming problems. Second, it is an extension of matrix row-reduction skills. Third, the Simplex Method can be used to illustrate the practicality of more advanced mathematics. An example using three independent variables would easily show the advantage of the Simplex Method over finding a solution through a graphical approach. In fact, the next example in class would be a system with three independent variables. Students would now notice the ease in which an optimal solution can be found, compared to graphing and solving for the intersections.

5 Appendix: Calculator Commands for Matrices

To enter a matrix, you need to go to `Matrix` (2nd function of the x^{-1} button) - `Edit`, and choose any matrix name (for example `[A]`). Once selected, you will need to enter the number of rows and columns. The calculator then shows you which entry needs input. When finished entering a matrix, press `2nd function` - `Mode` to quit editing.

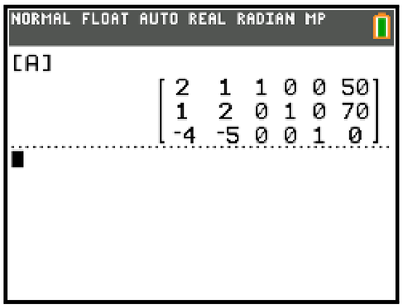
The following commands are found at the bottom of the `Matrix - Math` menu.

- `rref([A]` will reduce matrix [A] into reduced row-echelon form.
- `row+([A],1,2)` will add rows 1 and 2 in matrix [A] and create a new row 2.

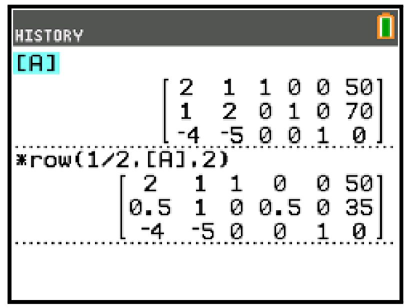
- `*row(1/2, [A], 1)` will multiply row 1 of matrix [A] by and create a new row 2.
- `*row+(-2,[A], 2,1)` will multiply row 2 of matrix [A] by -2, add it to row 1 and create a new row 1.

One common mistake made by students is referring to an incorrect matrix. Once a change is made in any matrix, the resulting matrix is stored in the answer memory location, `Ans`, found at `2nd function - negative`. Any resulting matrix can be saved as another matrix by calling up `Ans`, then pressing the button, and going to `Matrix - Names` and choosing a matrix. This can be done at any time.

Figures 7-9 the actual screen captures for this problem. Press `MATH` $\rightarrow$ `FRAC` to convert from decimals to fractions.

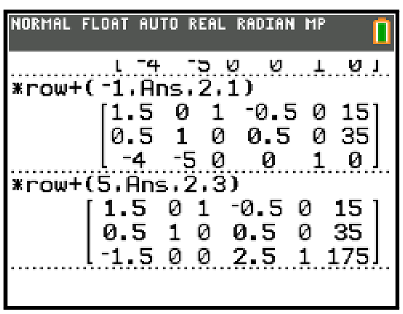


(a) *Displaying matrix [A].*

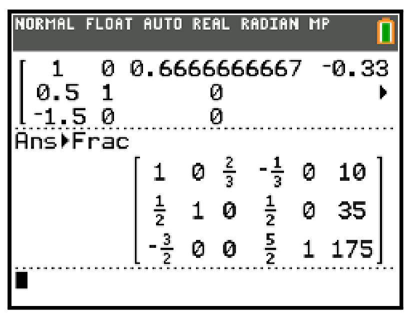


(b) *Multiplying row 2 by $\frac{1}{2}$.*

Fig. 7

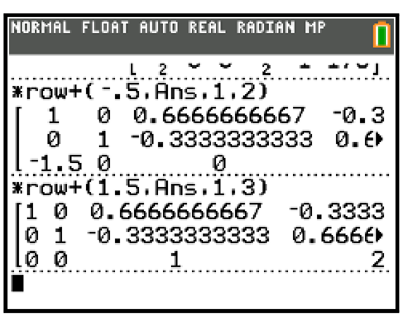


(a) *Various row operations.*

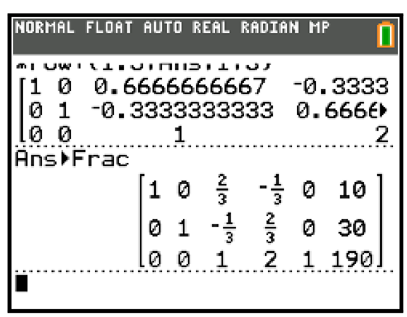


(b) *Multiplying row 2 by $\frac{1}{2}$.*

Fig. 8



(a) *Various row operations.*



(b) *Converting decimal entries to fractional form.*

Fig. 9

References

Common Core State Standards Initiative. (2011). *Common core state standards for mathematics*.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Todd A. Moyer, tmoyer@towson.edu, is an associate professor of Mathematics at Towson University. Before teaching at Towson, he taught high school mathematics for 15 years. He incorporates technology into all of his lessons.