A proposition from algebra: How would Pythagoras phrase it?

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Abstract: *The authors present three methods from different branches of mathematics to solve the same task. Originally an algebraic problem, the first solution presented utilizes an algebraic method. Subsequently, the authors translate the question into a geometric language and explore ways Pythagoras might phrase and solve the problem. Lastly, they share geometric and trigonometric solutions dealing with a special triangle whose angles are α, 2α, and 4α - a geometric progression. Solution of the same problem using different methods deepens student mathematical understanding while promoting mathematics as a field composed of intertwining branches.*

Keywords. *Rich tasks, problem solving, connections, proof*

1 Introduction

In recent times, much consideration has been given to problem solving and teaching methods that connect various areas of mathematics into a more unified whole. Educators in the field of mathematics agree that using more than one approach to solve the same problem promotes the development of mathematical reasoning (Polya, 1973; Schoenfeld, 1985; National Council of Teachers of Mathematics, 2000). The solution of problems using different methods encourages flexibility and creativity (Tall, 2007; Leiken & Lev, 2007). Moreover, proving a result or solving a problem using methods from various branches in mathematics (e.g., geometry, trigonometry, analytical geometry, vectors, complex numbers) deepens mathematical understanding. Our approach of presenting multiple proofs of the same problem as a tool for constructing mathematical connections is supported by Polya (1973, 1981), Schoenfeld (1988), Ersoz (2009), The National Council of Teachers of Mathematics (2000), and Levav-Waynberg and Leiken (2009).

Similar to the idea of "one problem with many solutions/proofs," is the idea of multiple solution tasks presented by Leiken & Lev (2007), Leiken (2009), Levav-Waynberg and Leiken (2009), and Stupel & Ben-Cahim(2013). Leiken (2009) points out that differences between proof methods may be explained along four dimensions, namely: (1) different representations of a mathematical concept; (2) different properties (definitions or theorems) of mathematical concepts from a particular mathematical topic; (3) mathematical tools and theorems from different mathematical branches;

(4) different theorems and tools from different subjects (not necessarily from mathematics). In our case, we apply the third type of difference: in this paper, we present different solutions using tools and theorems from Euclidean geometry, analytical geometry, trigonometry, vectors, and complex numbers. We present a geometric proof for an algebraic proposition that students typically prove using algebraic techniques. Our geometric proof is composed of four auxiliary propositions.

2 The proposition

If a, b, and c are positive numbers, and (1) and (2) hold:

$$a^2 + ac = b^2 \tag{1}$$
$$b^2 + ab = c^2 \tag{2}$$

then we may conclude

$$a^2 + bc = c^2 \tag{3}$$

3 An algebraic solution

The solution presented here is one of several possible methods accessible to students in early secondary level courses (i.e., typical 14-15 year olds).

From (1) we eliminate and obtain

$$c = \frac{b^2 - a^2}{a} \tag{4}$$

Next, we substitute the resulting expression for c in (2), and obtain

$$b^2 + ab = \frac{\left(b^2 - a^2\right)^2}{a^2}$$
$$b(a + b) = \frac{(b - a)^2(b + a)^2}{a^2}$$

Since a and b are positive, $b + a \neq 0$, therefore $b = \frac{(b-a)^2(b+a)}{a^2}$.

We expand, collect similar terms, and obtain

$$b^3 - b^2a - 2a^2b + a^3 = 0 \tag{5}$$

We denote $a^2 + bc - c^2$ as M and prove that $M = 0$ (which proves the proposition).

We substitute $\frac{b^2-a^2}{a}$ for c from (4), and $b^2 + ab$ for c^2 from (2) into the expression for M, obtaining

$$M = a^2 + \frac{b(b^2 - a^2)}{a} - (b^2 + ab)$$

From (5) we obtain: $M = \frac{a^3+b^3-a^2b-b^2a-a^2b}{a} = \frac{a^3-2a^2b-b^2a+b^3}{a} = 0$ and, hence, it follows that $a^2 + bc = c^2$.

4 How would Pythagoras have formulated the problem, had he lived today?

Now we recast the problem in a geometric context, asking how Pythagoras may have formulated the proposition. In ancient Greece, a^2 meant the area of the square whose side length is a, and bc meant the area of the rectangle whose side lengths are b and c. Therefore, is probable that Pythagoras would have given the problem a geometrical interpretation such as the following.

4.1 The proposition in its geometrical form

Recasting the original proposition in a geometric context, we are given three segments whose lengths are a, b, c, as shown in Figure 1.

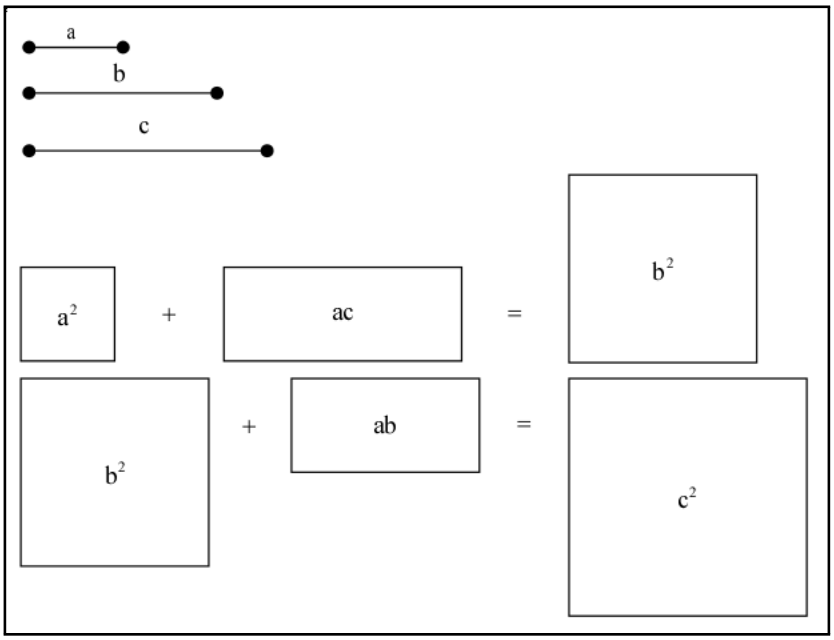

Fig. 1: *Geometric interpretation of the original proposition.*

In a corresponding manner, we prove that the geometric equivalent holds, as shown in Figure 2.

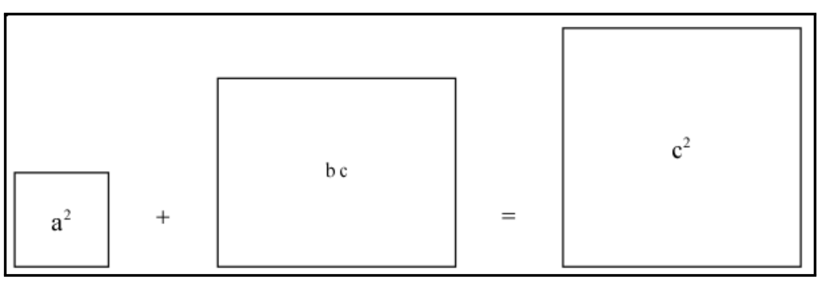

Fig. 2: *Geometric interpretation of the original result.*

Note that the notation inside the squares is modern script. The Greeks would likely have only used the drawings of the quadrilaterals.

5 A geometric proof

The proof we offer here is comprised of 4 auxiliary propositions (lemmas), after which we present a proof of the main proposition.

Lemma 5.1. *The following inequality holds: $c > b > a$*

Proof. From (1) it follows that $b > a$, and from (2) it follows that $c > b$. □

Lemma 5.2. *The following inequality holds: $a + b > c$.*

Proof. It is given in (2) that $b^2 + ab = c^2$. From proposition (1) it follows that $c > b$, therefore it is clear that $a + b > c$, because $(a + b)b = c^2$. From the segments a, b, c, one can form a triangle (by combining propositions (1) and (2)). □

Lemma 5.3. *If in the triangle ABC, whose sides are a, b, c, and whose angles are α, β, and γ, there holds (1) $a^2 + ac = b^2$, then $\beta = 2\alpha$.*

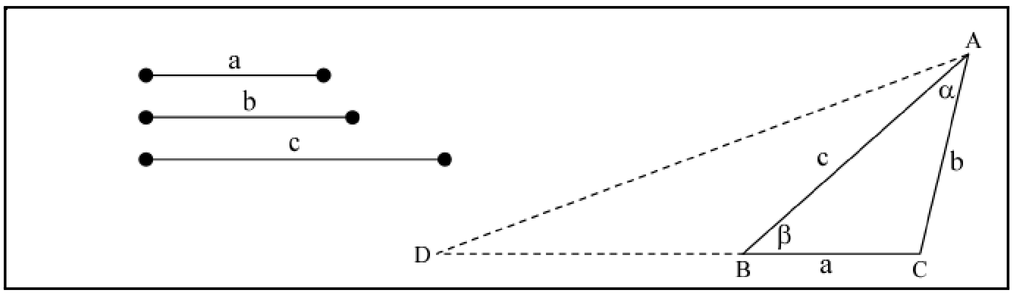

Fig. 3: *Geometric interpretation of Lemma 5.3.*

Proof. We extend CB to D, so that $BD = AB$ (see Figure 3).Therefore $DC = a + c$. But from the data there holds $a(a + c) = b^2$, therefore $\frac{a+c}{b} = \frac{b}{a}$, in other words $\frac{DC}{AC} = \frac{AC}{BC}$. Therefore, from the first theorem of similarity, $\triangle ABC \sim \triangle DAC$, and therefore $\angle D = \alpha$. But $DB = AB$ (the auxiliary construction), and hence $\beta = 2\alpha$. □

Lemma 5.4. *If in the triangle ABC whose sides are a, b, c, and whose angles are α, β, and γ, there holds $a^2 + ac = b^2$ and $b^2 + ab = c^2$, then $\gamma = 4\alpha$ (see Figure 4).*

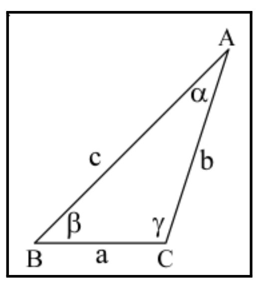

Fig. 4: *Geometric interpretation of Lemma 5.4.*

Proof. From Lemma 5.3, $\beta = 2\alpha$. We extend AC to E, so that $BC = CE = a$ (see Figure 5). From the data, there holds $b^2 + ab = c^2$, therefore $b(a + b) = c^2$. and hence it follows that $\frac{b}{c} = \frac{c}{a+b}$. In the same manner (as in the proof of Lemma 5.3), we have $\triangle ABC \sim \triangle AEB$. Therefore, $\angle E = 2\alpha$, but the triangle BCE is an isosceles triangle, and hence $\gamma = 4\alpha$. □

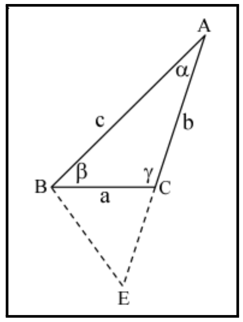

Fig. 5: *Extending AC to E.*

5.1 Conclusion from Lemmas 5.1 - 5.4

If a, b, and c are three segments that statisfy the relations $a^2 + ac = b^2$ and $b^2 + ab = c^2$, then from the segments a, b, and c, one can form $\triangle ABC$ with magnitudes of angles $\angle A$, $\angle B$, and $\angle C$ equal to α, 2α, and 4α, respectively. Therefore, $7\alpha = 180°$, and $\alpha = \frac{180°}{7}$ (see Figure 6).

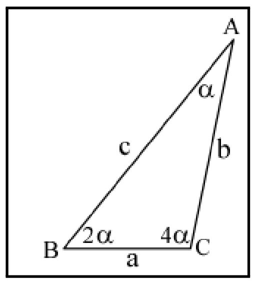

Fig. 6: *Angle measures in $\triangle ABC$.*

We now return to the main proposition and prove it.

Theorem 5.5. *To show:*

$$\begin{aligned} a^2 + ac &= b^2 \\ b^2 + ab &= c^2 \end{aligned} \qquad \Longrightarrow \qquad a^2 + bc = c^2$$

Proof. See Figure 7. We mark off the segment AD on AB, so that, $AD = AC = b(c > b)$. Therefore the length of the segment BD is $c - b$. The triangle ADC is isosceles with vertex angle of α, and therefore each of the base angles is 3α (since it was proved that $\alpha = \frac{180°}{7}$), and also $\angle C = 4\alpha$, and therefore $\angle DCB = \alpha$. Hence, it follows that $\triangle BCD \sim \triangle BAC$, and there holds $BC^2 = BD \cdot BA$, in other words $a^2 = (c - b)c$, and it follows that $a^2 + bc = c^2$. $\square$

6 A trigonometric proof

In order to present mathematics as an extensive field comprised of various interconnected branches, we present a trigonometric proof of the task.

Theorem 6.1. *Given is $a, b, c > 0$ and*

$$a^2 + ac = b^2 \tag{6}$$

$$b^2 + ab = c^2 \tag{7}$$

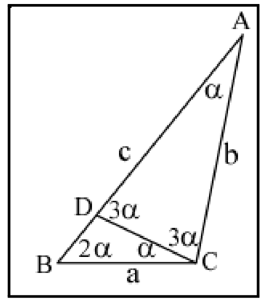

Fig. 7: *Visual confirmation of the relationship.*

We prove the following.

$$a^2 + bc = c^2 \tag{8}$$

Proof. As in the geometric proof, we note that $c > a > b$ and $a + b > c$ and therefore the lengths of the segments a, b, and c represent segments that form a triangle.

From the Law of Sines, we obtain from (6) that

$$sin^2(\angle A) + sin(\angle A) \cdot sin(\angle C) = sin^2(\angle B) \implies \tag{9}$$

$$sin(\angle A) \cdot sin(\angle C) = sin^2(\angle B) - sin^2(\angle A) \tag{10}$$

$$= (sin(\angle B) - sin(\angle A))(sin(\angle B) + sin(\angle A)) \tag{11}$$

Using the trigonometric identities for the sum and difference of angles, we obtain the following.

$$2sin\frac{\angle B - \angle A}{2} \cdot cos\frac{\angle B + \angle A}{2} \cdot 2sin\frac{\angle B + \angle A}{2} \cdot cos\frac{\angle B - \angle A}{2} = sin(\angle A + \angle B) \cdot sin(\angle B - \angle A) \tag{12}$$

$$= sin\angle C \cdot sin(\angle B - \angle A) \tag{13}$$

Since $sin\angle C \neq 0$, we obtain.

$$sin\angle A = sin(\angle B - \angle A) \tag{14}$$

Since this is a triangle, we have $\angle A = \angle B - \angle A \implies \angle B = 2\angle A$. In the same manner, from (7) we obtain $\angle C = 2\angle B$. From the relations obtained between the angles, we have $\angle C = 2\angle B = 4\angle A$, and since the sum of angles in the triangle is $\angle A + \angle B + \angle C = 180°$, we obtain

$$\angle A + 2\angle A + 4\angle A = 7\angle A = 180° \implies \angle C = \frac{180° + \angle A}{2} \tag{15}$$

$$2\angle C = 180° + \angle A \implies \angle C - \angle A = 180° - \angle C \tag{16}$$

From which there follows:

$$sin(\angle C - \angle A) \cdot sin\angle B \implies sin\angle B \cdot sin\angle C \tag{17}$$

And therefore

$$sin^2\angle C - sin^2\angle A = sin\angle B \cdot sin\angle C \implies a^2 + bc = c^2 \tag{18}$$

□

7 Concluding remarks and implications for teaching

We constructed proofs of our proposition using a combination of several fields in mathematics. In our work, geometry came to the aid proof in algebra while algebra provided the source of our challenging problem. We used trigonometry to construct another proof. Through it all, it was apparent to us how beautiful mathematics is. In the words of Hardy (1940), "the mathematician's patterns, like the painter's or the poet, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics." The three different methods we employed resulted in the same relations between the sides of the triangle. Geometry is a goldmine for multiple solution tasks. Proofs may be derived by applying different methods within the specific topic of geometry or within other mathematical areas. The multiple solutions that were presented herein for one problem demonstrate the connectivity between different areas of mathematics. Multiple proofs foster both better comprehension and increased creativity in mathematics for the student/learner and challenge for the teacher.

References

Ersoz, F. A. (2009). Proof in different mathematical domains. *ICME Study, 1*, 160–165.

Hardy, G. H. (1940). *A mathematician's apology*. Cambridge, UK: Cambridge University Press, 40.

Levav-Waynberg, A., & Leikin, R. (2009, January). Multiple solutions for a problem: A tool for evaluation of mathematical thinking in geometry. In *Proceedings of CERME* (Vol. 6, pp. 776–785).

Leikin, R., & Lev, M. (2007, July). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. In *Proceedings of the 31st international conference for the psychology of mathematics education* (Vol. 3, pp. 161–168). Seoul, Korea: The Korea Society of Educational Studies in Mathematics.

Leikin, R. (2009). Multiple proof tasks: Teacher practice and teacher education. *ICME Study, 19*(**2**), 31–36.

Leikin, R., & Lev, M. (2007, July). Multiple solution tasks as a magnifying glass for observation of mathematical creativity. In *Proceedings of the 31st international conference for the psychology of mathematics education* (Vol. 3, pp. 161–168). Seoul, Korea: The Korea Society of Educational Studies in Mathematics.

National Council of Teachers of Mathematics (Ed.). (2000). *Principles and standards for school mathematics* (Vol. 1). Reston, VA: Author.

Polya, G. (2014). *How to solve it: A new aspect of mathematical method.* Princeton University Press.

Schoenfeld, A. H. (1985). Metacognitive and epistemological issues in mathematical understanding. *Teaching and learning mathematical problem solving: Multiple research perspectives*, 361–380.

Schoenfeld, A. H. (1988). When good teaching leads to bad results: The disasters of 'well-taught' mathematics courses. *Educational Psychologist, 23*(**2**), 145–166.

Stupel, M., & Ben-Chaim, D. (2013). One problem, multiple solutions: How multiple proofs can connect several areas of mathematics. *Far East Journal of Mathematical Education, 11*(**2**), 129.

Tall, D. (2007, March). Teachers as Mentors to encourage both power and simplicity in active mathematical learning. In *Plenary at The Third Annual Conference for Middle East Teachers of Science, Mathematics and Computing* (Vol. 3, pp. 17–19).

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