Examples of multiple proofs in geometry: Part 1, tasks and hints

Ruth Segal, *Shaanan Academic Religious Teachers' College, Haifa, Israel*
Moshe Stupel, *Shaanan Academic Religious Teachers' College, Haifa, Israel*
Alfinio Flores, *University of Delaware*

Abstract: *Use of multiple methods for proof problems in geometry has great potential to expand reasoning and thinking skills. Discovering or understanding different solutions may develop and increase the enjoyment found in problem solving and render mathematics a creative enterprise. Six proof problems from geometry are presented with hints for multiple methods of proof.*

Keywords. *Geometry, proof, problem-solving*

1 Introduction

1.1 The value of multiple solutions in mathematics

One way to enhance the understanding of mathematical concepts and principles is to find a variety of solution methods for a single problem. In the same way that we might get a better perception of an object by using two senses, we might get a better understanding of a problem by using two proofs, which may give us further understanding or allow us to look at the problem from a different perspective (Polya, 1973). Minsky said that "You don't understand anything until you learn it more than one way" (quoted by Herold, 2005, p. 101). Forging connections between different areas of mathematics and showing multiple paths to the solution of a problem is a key part of developing of mathematical reasoning (Polya, 1973; Schoenfeld, 1985). Teaching standards in several countries recommend that students be exposed to multiple procedures and encouraged to compare methods, in order to develop flexibility in problem solving (Rittle-Johnson, Star, & Durkin, 2012). Furthermore, if students see mathematics problems as having a single solution method, they will see mathematics as a rigid discipline - not as a creative enterprise (Rigelman, 2013).

1.2 Plane geometry - Integration of fields and methods of solution

Plane geometry is among the most astounding branches of mathematics, due to the multitude of solution methods that exist. Indeed, there are hundreds of different proofs of the Pythagorean Theorem (Loomis, 1968). Use of auxiliary constructions can offer different methods of solution for the same task, and knowledge of advanced theorems gives rise to short and direct proofs. When students have multiple methods for proving and understanding proof exercises in geometry, they develop confidence, interest, ability, and flexibility in problem solving (Jiang & O'Brien, 2012). With this in mind, we showcase six proof exercises from plane geometry. For each problem we provide questions and hints to guide students to find multiple methods of proof. Some methods are standard, and others are unusual or very short and, perhaps, beautiful. The last exercise is more difficult. Our hope is to provide a set of examples with a wide range of proofs to encourage creativity, wonder and joy in the geometry classroom. An individual, small group or whole class project might be to find as many proofs as possible for each statement, encouraging novel proofs that are not indicated by guided questions. This would increase the joyful possibility that students might find a proof that was not previously considered by the teacher. Complete demonstrations can be found in Part 2, proofs at `http://www.math.udel.edu/~alfinio/mpp2.pdf`

Some of the approaches use advanced concepts that the students may not be aware of when they could first approach the problem. The teacher could either wait to present the problem until the multiple approaches could be understood, or introduce the problem early and then come back when the advanced results have been established. This would need some forethought in planning the course and would allow opportunity for review. The teacher may also want to do some combination of the above.

2 Examples of proof problems with multiple solutions

2.1 Problem 1

Given a right triangle ABC, $m\angle B = 90°$, $m\angle C = 30°$ and a point D on $\overline{BC}$ which divides $\overline{BC}$ so that $CD = 2BD$. Denote BD by x and DC by $2x$ (see Figure 1). Prove that $\overline{AD}$ bisects $\angle BAC$.

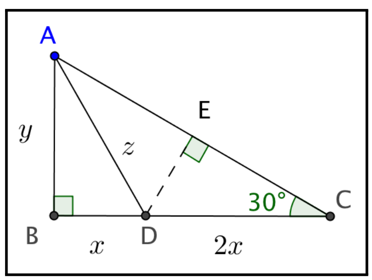

Fig. 1: *Trisecting a leg.*

2.1.1 Hints - method 1A

In a $30° - 60° - 90°$ right triangle the length of the hypotenuse is twice the length of the side opposite the $30°$ angle, i.e., $AC = 2y$. Apply the Pythagorean theorem to $\triangle ABC$ to obtain y in terms of x, and again to to compute z in terms of x. What can you say about $\angle BAD$?

2.1.2 Hints - method 1B

Construct perpendicular to the hypotenuse (dotted line in Figure 1). In the $30° - 60° - 90°$ triangle EDC, what can you say about DE compared to DC? Show triangles ABD and CED are congruent.

2.1.3 Hints - method 1C

The angle bisector divides the opposite side in the same ratio as the sides that form the angle (see Problem 4). Consider the ratios $\frac{AC}{AB}$ and $\frac{CD}{BD}$, and apply the converse of the angle bisector theorem.

2.1.4 Hints - method 1D

A $30° - 60° - 90°$ triangle is half of an equilateral triangle. What can you say about the ratio of segments of intersected medians in a triangle? In an equilateral triangle, how are angle bisectors and medians related?

2.2 Problem 2

Given the plane figure with $AB = AC = AD$ as shown in Figure 2. Prove $m\angle DBC = \frac{1}{2}\angle CAD$.

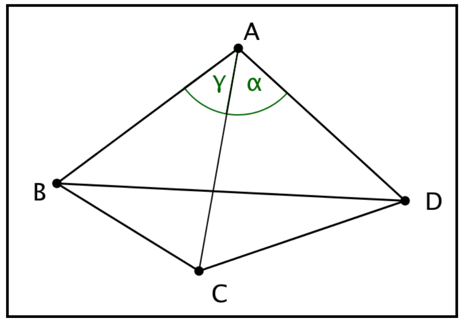

Fig. 2: *Three isosceles triangles.*

2.2.1 Hints - method 2A

Triangles ABC, CAD, and ABD are isosceles triangles; calculate the measures $m\angle ADC$, $m\angle ACB$, and $m\angle ABD$ in terms of α and γ. Express $m\angle DBC$ in terms of $m\angle ABC$ and $m\angle ABD$.

2.2.2 Hints - method 2B

Construct a circle with center A passing through points B, C, and D. Use the relation of inscribed and central angles subtending the same arc.

2.3 Problem 3

Given triangle ABC; $\overline{AD}$ is a bisector of the angle $\angle BAC$, and $AB > AC$. Prove $BD > DC$.

2.3.1 Hints - method 3A

See Figure 3. From $AB > AC$ it follows that $\gamma > \beta$. Add a point E on $\overline{AB}$ such that $AE = AC$. Show that $\triangle ADC \cong \triangle ADE$, $\angle ADE \cong \angle ADC$, $\angle AED \cong \angle ACD$, and $\overline{DE} \cong \overline{DC}$. We denote the measure of the first pair of angles by ϵ and observe that $m\angle AED = m\angle ACD = \gamma$. Denote $m\angle BED$ by δ. Show that $\epsilon > \beta$ and $\delta > \epsilon$, and therefore $\delta > \beta$. Relate the size of these angles in triangle BDE to the lengths BD and ED of corresponding opposite segments.

2.3.2 Hint - method 3B

Use the fact that $\frac{AB}{AC} = \frac{BD}{DC}$ (because $\overline{AD}$ is an angle bisector), and $AB > AC$.

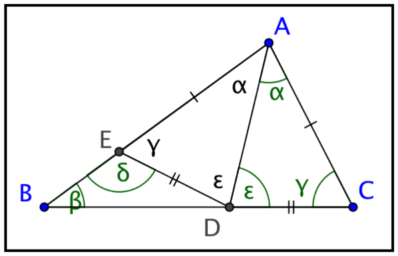

Fig. 3: *Auxiliary triangle.*

2.4 Problem 4 (Different proofs of the Angle Bisector Theorem)

Given $\triangle ABC$ and a point D on $\overline{AC}$ such that $\overline{BD}$ bisects $\angle ABC$, prove that $\overline{BD}$ divides $\overline{AC}$, such that $\frac{AD}{DC} = \frac{AB}{BC}$, i.e., prove that an angle bisector in a triangle divides the side it intersects into two segments such that the ratio of the lengths of the segments is equal to the ratio between the lengths of the enclosed angle. We denote the bisected angle by 2β.

2.4.1 Hints - method 4A

Through vertex A draw a line parallel to $\overline{BC}$. Through C draw a line parallel to angle bisector $\overline{BD}$ as shown in Figure 4. Show that triangle ABE is isosceles. Apply Thales' theorem to lines $\overline{AC}$ and $\overline{AF}$, intersected by parallel lines $\overline{DE}$ and $\overline{CF}$, to obtain $\frac{AD}{DC} = \frac{AE}{EF}$. Show that $EF = BC$ (because $BEFC$ is a parallelogram). Make two substitutions into the equality of ratios to obtain $\frac{AD}{DC} = \frac{AB}{BC}$.

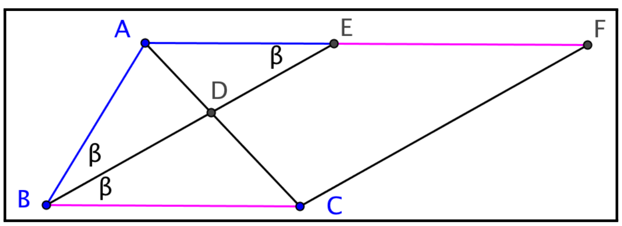

Fig. 4: *Auxiliary parallel lines.*

2.4.2 Hints - method 4B

Let E be the point where angle bisector $\overline{BD}$ intersects the circumcircle of $\triangle ABC$ (Figure 5). Show that $m\angle ACE = m\angle ABE = m\angle CBE = m\angle CAE = \beta$ and that $\triangle AEC$ is an isosceles triangle with $AE = CE$ (denote this length by t). Show that $\triangle BDC \sim \triangle ADE$ and therefore $\frac{BC}{t} = \frac{DC}{DE} \Longrightarrow t = \frac{BC \cdot DE}{DC}$. Similarly show that $t = \frac{AB \cdot DE}{AD}$ and $\frac{AB \cdot DE}{AD} = \frac{BC \cdot DE}{DC}$.

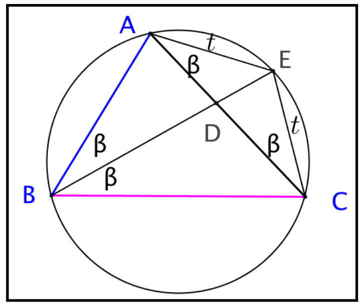

Fig. 5: *Angle bisector and circumcircle.*

2.4.3 Hints - method 4C

The ratio of the areas of two triangles with the same height is equal to the ratio of their bases. Using Figure 6, compare the ratio $\frac{\text{area}(\triangle ABD)}{\text{area}(\triangle ADC)}$ with the ratio of segment lengths BD and DC (use in the area formula the height of the triangle and segments $\overline{BD}$ and $\overline{DC}$), and with the ratio of side lengths AB and AC (use in the formula the dotted lines as height, and sides $\overline{AB}$ and $\overline{AC}$) (Figure 6).

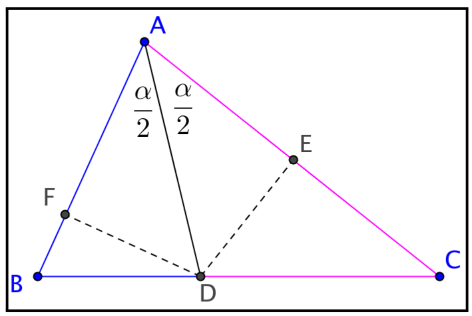

Fig. 6: *Equal heights.*

2.5 Problem 5

$ABCD$ is a parallelogram. The points E and F are the midpoints of the sides $\overline{AD}$ and $\overline{DC}$, respectively. We connect them with vertex B. The lines $\overleftrightarrow{BE}$ and $\overleftrightarrow{BF}$ intersect the diagonal $\overline{AC}$ at the points G and H, respectively. Prove that $AG = GH = HC$ (Figure 7).

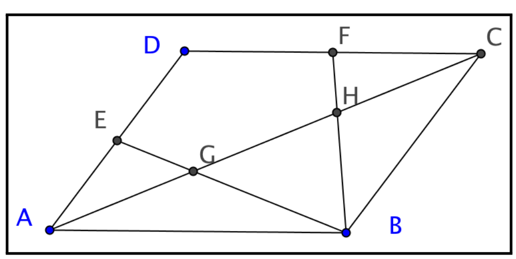

Fig. 7: *Trisecting the diagonal.*

2.5.1 Hints - method 5A

Draw the diagonal $\overline{BD}$. In triangle ABD consider the medians $\overline{BE}$ and $\overline{AI}$ to see that the segments have the relative lengths indicated in Figure 8 (y and $2y$).

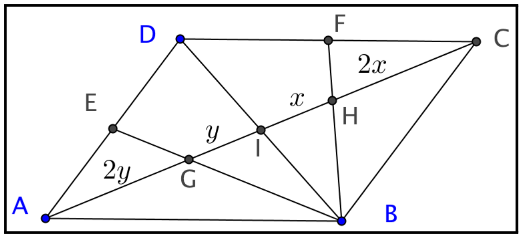

Fig. 8: *Two auxiliary triangles.*

Consider the medians $\overline{BF}$ and $\overline{CI}$ in triangle CBD. Use the fact that the diagonals in a parallelogram bisect each other to show that $x = y$.

2.5.2 Hints - method 5B

Connect the points D and G and extend the line to intersect side $\overline{AB}$ at point J (Figure 9). Show that J is the midpoint of side $\overline{AB}$. Show that $DJBF$ is a parallelogram. Use the relation between AJ and JB to show $AG = GH$. Similarly show that $GH = HC$.

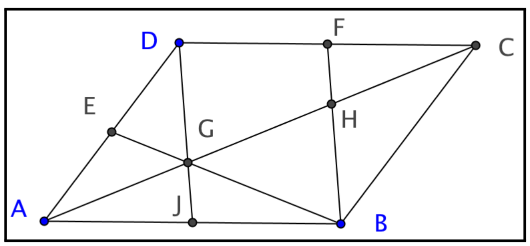

Fig. 9: *Another trisection.*

2.5.3 Hints - method 5C

From vertex C draw a line parallel to $\overline{BF}$, which intersects the continuation of the side $\overline{AB}$ at point K (Figure 10). The quadrilateral $FCKB$ is a parallelogram. Hence: $AJ = JB = BK$, because $\overline{GJ} \parallel \overline{GH} \parallel \overline{HC}$, and from the proven identity, $AJ = JB = BK$, as well as from Thales' theorem for triangle ACK we have that $AG = GH = HC$.

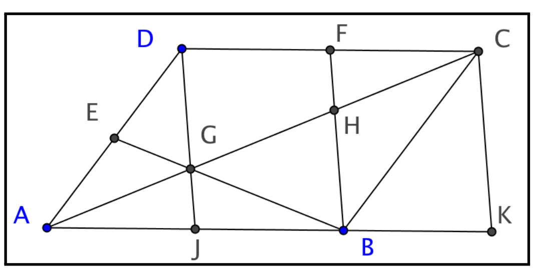

Fig. 10: *Auxiliary parallel line.*

2.6 Problem 6

Square $ABCD$ has side length a. E is a point on side between D and C. Through vertex A draw a straight line that intersects the side at point E. Through vertex A draw the bisector of angle EAB which intersects the side $\overline{BC}$ at point F. Prove that $x + y = z$ ($BF + DE = AE$) (Figure 11).

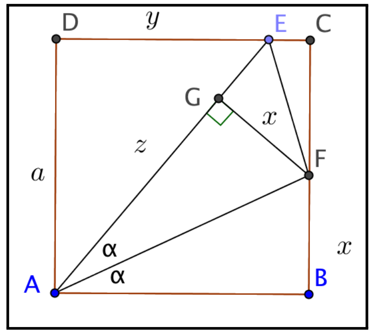

Fig. 11: $x + y = z$.

2.6.1 Hints - method 6A

From point F drop a perpendicular $\overline{FG}$ to the segment $\overline{AE}$. Why is $BF = FG = x$? Calculate the area of the square $ABCD$ from the areas of the four triangles ABF, FCE, EFA, and ADE. Simplify to obtain an expression of a^2 in terms of x, y, and z. Use the Pythagorean theorem in triangle ADE to obtain a^2 in terms of z and y.

2.6.2 Hints - method 6B

From point E drop a perpendicular $\overline{EN}$ to the side $\overline{AB}$ (see Figure 12). Show that $\frac{z}{y} = \frac{EM}{MN}(1)$. Use the fact that triangles AMN and AFB are similar to express MN in terms of x, y, and a. Express $EM = a - MN$ in terms of x, y and a. Substitute these expressions for EM and MN in proportion (1) to express a^2 in terms of x, y and z. Use the Pythagorean theorem in triangle ADE.

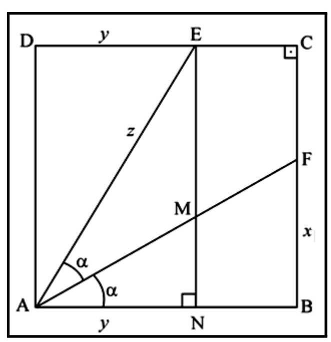

Fig. 12: *Auxiliary parallel line.*

2.6.3 Hints - method 6C

Rotate triangle ABF $90°$ around A so that side $\overline{AB}$ coincides with side $\overline{AD}$ (Figure 13). Show that triangle $F'EA$ is isosceles.

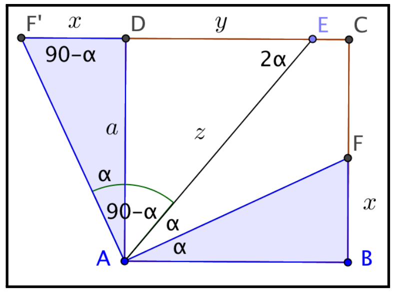

Fig. 13: *Rotating a triangle.*

2.6.4 Hints - method 6D

$A(0,0)$, $B(a,0)$, $C(a,a)$, and $D(0,a)$ are the coordinates of the vertices. Choose a point $E(b,a)$ $(b<a)$ on the side $\overline{DC}$. Find the equations of line l_1 containing $\overline{AB}$ and line l_2 containing $\overline{AE}$ in general form $(ax+by+c=0)$. For two straight lines with general equations $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ the equations of the angle bisectors are $\frac{a_1x+b_1y+c_1}{\sqrt{{a_1}^2+{b_1}^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{{a_2}^2+{b_2}^2}}$. Obtain the equation of the angle bisector of lines l_1 and l_2 that has positive slope to find the coordinates of point F. Obtain the lengths of the segments $\overline{BF}$, $\overline{AE}$ and $\overline{DE}$ and show that $BF+DE=AE$.

3 Final remarks

The tasks presented are a small part of a wide range of proof problems in geometry for which more than one method of proof can be found. The range of solutions illustrates the role of creativity in mathematics, and encourages and challenges one to find additional solutions, which are sometimes surprising and unusual. These solutions provide skills, tools and methods of solution, and they allow one to deal with difficult tasks, which contribute to the development of reasoning and which cause much joy to the fans of mathematics.

References

Herold, R. (2005). *Managing an information security and privacy awareness and training program*. Boca Raton, FL: CRC Press.

Jiang, Z., & O'Brien, G. E. (2012). Multiple proof approaches and mathematical connections. *Mathematics Teacher, 105*(**8**), 586-593.

Loomis, E. S. (1968). *The Pythagorean proposition*. Washington, DC: National Council of Teachers of Mathematics.

Polya, G. (1973). *How to solve it: A new aspect of mathematical method*. (2nd edition). Princeton, NJ: Princeton University Press.

Rigelman, N. R. (2013). Becoming a mathematical problem solver. *Mathematics Teaching in the Middle School, 18*(**7**), 416-423.

Rittle-Johnson, B., Star, J. R., & Durkin, K. (2012). Developing procedural flexibility: Are novices prepared to learn from comparing procedures? *British Journal of Educational Psychology, 82*(**3**), 436-455.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando, FL: Academic Press.

Ruti Segal, rutisegal@gmail.com, taught high school mathematics for 26 years. She became involved in teacher training and teaches mathematics and didactic of mathematics focusing on junior and senior high school Curriculum at Shaanan College and Oranim College, in Israel. She integrates technology in mathematics instruction and research and engages in public service as a mathematics assistant to the superintendent at the senior high school.

Moshe Stupel, stupel@bezeqint.net, is a professor and a pre-service teacher educator in two academic colleges, Shaanan Academic College and Gordon College. He has published and presented 40 papers on mathematics and mathematics education. Recently, his research is focused on various methods of problem solving and on variance and invariance property in dynamic geometrical environments (DGEs) in mathematics.

Alfinio Flores, alfinio@udel.edu, teaches mathematics and mathematics methods courses *con ganas* at the University of Delaware. He uses technology to provide opportunities for students and teachers to explore mathematics on their own. He has conducted activities for students in schools ranging for Kindergarten to 12th grade, and conducted professional development sessions for teachers of grades K-16.