Exploring Sequences through Variations on Fibonacci

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Abstract: *Leonardo Pisano's word problem, which led to the ubiquitous Fibonacci sequence, provides a point of departure for several explorations. The variations are motivated by more realistic assumptions to the original rabbit problem. Through these activities, students practice data generation, pattern-finding, developing formal proofs, carrying out generalizations, and data analysis. And they might even have some fun!*
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1 Introduction

$1, 1, 2, 3, 5, 8, 13, \ldots$. Most textbooks, regardless of level, mention the famous Fibonacci sequence. Problems ask the student to find the recursive definition: "In the Fibonacci sequence 1, 1, 2, 3, 5, 8, . . ., you can find each term (after the first two) by adding certain terms together. Find the pattern and use it to write the next three terms of the sequence" (Chapin, Illingsworth, Landau, Masingila & McCracken, 2001, p. 487). Questions also explore patterns among the Fibonacci numbers: "Notice that the third term is divisible by 2. Are the 6th and 9th terms also divisible by 2? What conclusion can you draw about every third term? Why is this true?" (Burger et al., 2011, p. 280). Some texts use marginal notes to connect Fibonacci numbers to the number of petals on flowers (Burger et al., 2011, p. 280), the shape of nautilus shells (Stewart, Redlin & Watson, 2012, p. 788) and the golden mean (Burger et al., 2007, p. 866). A few textbooks highlight the origin of the sequence: "In 1202, Italian mathematician Leonardo Fibonacci described how fast rabbits breed under ideal circumstances. Fibonacci noted the number of pairs of rabbits each month and formed a famous pattern called the Fibonacci sequence" (Burger et al., 2007, p. 862).

Knowing that my college math students had seen the Fibonacci sequence in too many classes, for a math club activity I thought they might appreciate knowing the history of the sequence and enjoy exacting a small bit of revenge by messing with the assumptions of the original rabbit problem. Based on our success, the origin of this famous sequence can provide an interesting point of departure for student explorations that incorporate pattern-finding, formal proof, generalization, and even data analysis.

2 Fibonacci's Rabbits

Leonardo Pisano, also known as Fibonacci, gave a word problem in his book, *Liber Abaci*, that begins, "A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also" (Pisano, 2002, p. 404). It then proceeds to derive how many pairs there are each month for one year. His simple rules for reproduction are: (1) each month every adult pair gives birth to a new-born pair the next month, and (2) every new-born pair becomes an adult pair the next month.

Like most word problems, these simple assumptions are obviously unrealistic; we should ask the question, "What do we get if we change the assumptions?" At the math club we worked through the generations of rabbits with Fibonacci's rules, then did it again with an addition condition on how long each rabbit lived. While we did the exploration with pen and paper, we likely would have been quicker to see patterns using a spreadsheet. Keeping track of the young and adult rabbits separately is the key to discovering and proving the new sequence formulas.

Table 1: *Fibonacci rabbit patterns.*

Month	Adult Pairs	Young Pairs	Total
1	0	1	1
2	1	0	1
3	1	1	2
4	2	1	3
5	3	2	5
6	5	3	8
7	8	5	13
8	13	8	21
9	21	13	34
10	34	21	55

3 First Exploration

In order to count pairs of rabbits in each month, we must understand how they reproduce. Following Fibonacci's rule, if there is a new-born pair in one month, then it becomes an adult pair in the next month and gives birth to a new-born pair in the following month and each month after that. This first exploration leads students through generating the sequence of monthly populations for Fibonacci's problem, finding a generating formula, proving the formula, and then extending the idea by working with a variation to the litter size.

1. Starting with one new-born pair the first month, determine how many pairs of rabbits there will be each month. *Solution*: using a spreadsheet with columns for young and adult pairs and total pairs and a row for each month, it is easy to keep track. Starting with one young pair, the sequence of monthly populations is the familiar $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$.

2. Find a formula that describes how the population each month relates to the populations of the previous months. *Solution*: using F_n for the number of pairs in month n, the monthly

populations follow the familiar recursive formula $F_n = F_{n-1} + F_{n-2}$, but note that the pattern could also be described by $F_n = F_{n-1} + F_{n-3} + F_{n-4}$ or $F_n = 2F_{n-2} + F_{n-3}$.

3. Find the relationships between both the numbers of new-born pairs and of adult pairs each month and the respective numbers the previous month. Use these to prove the formula for the overall monthly populations found above. *Solution*: the number of young pairs one month equals the number of adult pairs the previous month while the number of adult pairs one month is the total of adult and young pairs the previous month. Thus, using b_n and R_n for the monthly populations of pairs of bunnies and rabbits, respectively, we see that $b_n = R_{n-1}$ and $R_n = R_{n-1} + b_{n-1}$. Then $F_n = R_n + b_n = R_{n-1} + b_{n-1} + R_{n-1} = F_{n-1} + R_{n-2} + b_{n-2} = F_{n-1} + F_{n-2}$.

4. (Extension) Assume that each pair of adult rabbits gives birth to 2 pairs of rabbits instead. Find, and prove, a relationship between the monthly populations of pairs of rabbits. What is the formula if they always give birth to K pairs? *Solution*: a similar process as above yields $F_n = F_{n-1} + 2F_{n-2}$ and $F_n = F_{n-1} + KF_{n-2}$. This is the "fertile Fibonacci" variation. The population numbers can be generated easily within a spreadsheet using the formulas $b_n = KR_{n-1}$ and $R_n = R_{n-1} + b_{n-1}$ for any specific K value.

4 Second Exploration

Since real rabbits do not continue to reproduce indefinitely, assuming a finite lifespan is natural. We use a short lifespan for the sake of easier computations, but any period can be used.

1. If all the pairs of rabbits die after their fifth month, after giving birth to their fourth pair of bunnies, find the monthly populations of pairs of rabbits. *Solution*: to the previous spreadsheet a column for the pairs who die each month can be added. The monthly populations are the same until the sixth month. The sequence is $1, 1, 2, 3, 5, 7, 11, 17, 26, \ldots$.

2. Find a formula that describes how the population of pairs each month relates to the populations the previous months. *Solution*: Both $F_n = F_{n-1} + F_{n-2} - F_{n-6}$ and $F_n = F_{n-2} + F_{n-3} + F_{n-4} + F_{n-5}$ work, but the former follows the expected general form: `current population = previous population + change`. We should set $F_0 = 1$ and all $F_{-k} = 0$ as the starting conditions. This is the "Fibonacci with death" variation.

3. Find formulas relating the monthly numbers of adult pairs and new-born pairs, and use them to prove the formula you found for the total monthly populations. *Solution*: $R_n = R_{n-1} + b_{n-1} - b_{n-5} = F_{n-1} - b_{n-5}$ includes the death condition while $b_n = R_{n-1}$ corresponds to births. Then, $F_n = R_n + b_n = F_{n-1} - b_{n-5} + R_{n-1} = F_{n-1} - R_{n-6} + F_{n-2} - b_{n-6} = F_{n-1} + F_{n-2} - F_{n-6}$. Alternatively, since the rabbits eventually die, the current population is also equal to the sum of all the bunnies born who have not died yet. Namely, $F_n = b_n + b_{n-1} + b_{n-2} + b_{n-3} + b_{n-4} = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4} + R_{n-5} = F_{n-2} + F_{n-3} + F_{n-4} + F_{n-5} + F_{n-6} - (b_{n-2} + b_{n-3} + b_{n-4} + b_{n-5} + b_{n-6}) = F_{n-2} + F_{n-3} + F_{n-4} + F_{n-5} + F_{n-6} - F_{n-6} = F_{n-2} + F_{n-3} + F_{n-4} + F_{n-5}$.

4. (Extensions) (a) Generalize the formula to a lifespan of any $K > 2$ months; (b) If new-born rabbits mature after two months, rather than one, then their offspring start appearing in the fourth month. If rabbits die after their fifth month, find the populations each month and appropriate formulas; (c) Generalize these for maturity after L months and death after $K > L$ months. *Solutions*: (a) $F_n = F_{n-1} + F_{n-2} - F_{n-1-K}$. (b) For delayed maturity, we must keep track of pairs of bunnies of each age (separate columns in the spreadsheet), such as $F_n = R_n + b_{(1)n} + b_{(2)n}$. Then $b_{(1)n} = R_{n-1}$, $b_{(2)n} = b_{(1)n-1}$ and $R_n = R_{n-1} + b_{(2)n-1} - b_{(1)n-5}$ yield $F_n = F_{n-1} + F_{n-3} - F_{n-6}$. (c) In general, $F_n = F_{n-1} + F_{n-1-L} - F_{n-1-K}$. This is the "mortal teenage Fibonacci" variation.

5 Third Exploration

A constant litter size is also unrealistic; hence, this variation introduces randomness in the births. Here the goal shifts to looking for apparent patterns in randomly generated data. Students can work through this exploration without having completed the proofs in the previous activities. For simplicity we assume no rabbits die.

1. If each adult pair of Fibonacci's rabbits gives birth to either no pairs or one pair each month, with the outcome determined randomly by a coin toss (heads = one pair, tails = no pairs), generate monthly populations of pairs of rabbits through the 10th month. *Solution*: the total number of young pairs the next month can be generated by tossing a number of coins equal to the number of adult pairs and counting the number of heads or by using a random binomial function found on many calculators. Each student will get a different sequence of monthly populations. The minimum monthly populations are all 1 (no heads are ever tossed) while the maximum monthly population is the corresponding Fibonacci number (only heads tossed). This is the "random Fibonacci" variation.

2. Repeat the process in #1 20 times and for each month find the mean, median, range, and standard deviation of the populations you generated, and graph each of the distributions. Solutions vary but for earlier months the distributions should appear quite regular, whereas for later months, 20 trials may not provide enough data points to assure a nicely shaped distribution (see Figure 1).

3. Consider the average monthly populations and make a conjecture about a relationship between them. *Solution*: it appears that the averages follow $A_n = A_{n-1} + .5A_{n-2}$. This makes sense if we recognize that on average each litter has .5 young pairs.

4. (Extension) Repeat this exploration with litter sizes determined by other probability devices, such as rolling a die, twirling a spinner, or tossing multiple coins.

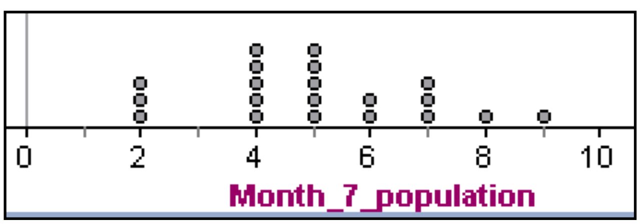

Fig. 1: *Distribution of month 7 population.*

6 Conclusion

The Fibonacci sequence is well known but its origin is not. Starting from that story problem and imposing variations provides motivation to discover and prove new relationships. My students enjoyed it so much they created a T-shirt to commemorate our (not actually original) discovery (see Figure 2). We later found mortality and delayed maturity assumptions discussed in the literature (Alfred, 1963; Hoggatt & Lind, 1969; Oller-Marcén, 2009). Apparently, however, the study of distributions of random Fibonacci bunnies is new. Imaginative students may suggest other variations (zombies rabbits come back to life and eat the young?) on Fibonacci's story that can lead to their own discoveries.

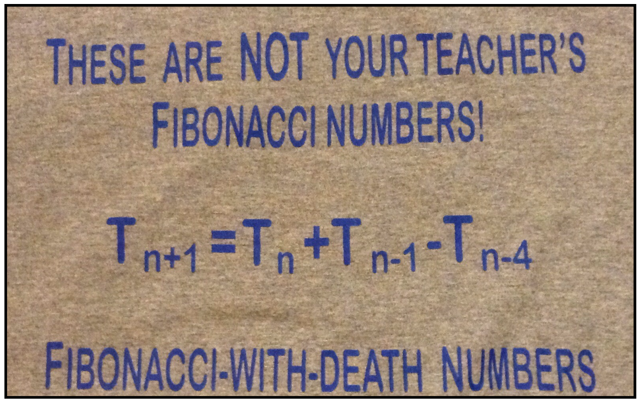

Fig. 2: *Fibonacci t-shirt design.*

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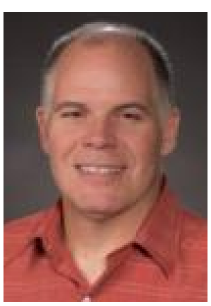

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