Productive Mathematical Discussions in Teaching Through Problem Solving

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Abstract: *This paper focuses on the importance of mathematical discussions and how these discussions can be conducted during teaching mathematics through problem-solving approach. In teaching through problem solving, the primary aim is to develop students' reasoning and connect students' solution strategies with new mathematical ideas. To enhance these types of discussions, mathematics teachers need to apply sociomathematical norms, make specific talk moves, and plan for discussions in advance.*

Keywords: *Discourse, assessment, problem solving*

1 Introduction

Mathematical discussion is much more than merely getting students to talk. What matters is discussions that develop students' reasoning and abilities to clearly present their mathematical ideas. Students learn when they think, talk, reason, and inquire (NCTM, 2014). This is what an approach to teaching called Teaching Through Problem Solving (TTPS) aims to do. Mathematical discussion plays a key role when implementing this type of teaching. This article will describe what a TTPS instructional approach entails and highlight the critical role discussions play in the learning process.

1.1 What is Teaching Through Problem Solving?

In this approach to teaching, the teacher begins a lesson with the goal of using an open-ended problem to connect students' solutions with a new mathematical idea (Schroeder & Lester, 1989; Stein et al., 2008). The teachers' role is to orchestrate mathematical discussion that uses students' solutions as a pathway to learn new mathematics. When using a TTPS approach, problem solving is the medium through which learning mathematics occur. Figure 1 shows the sequence used with the TTPS approach.

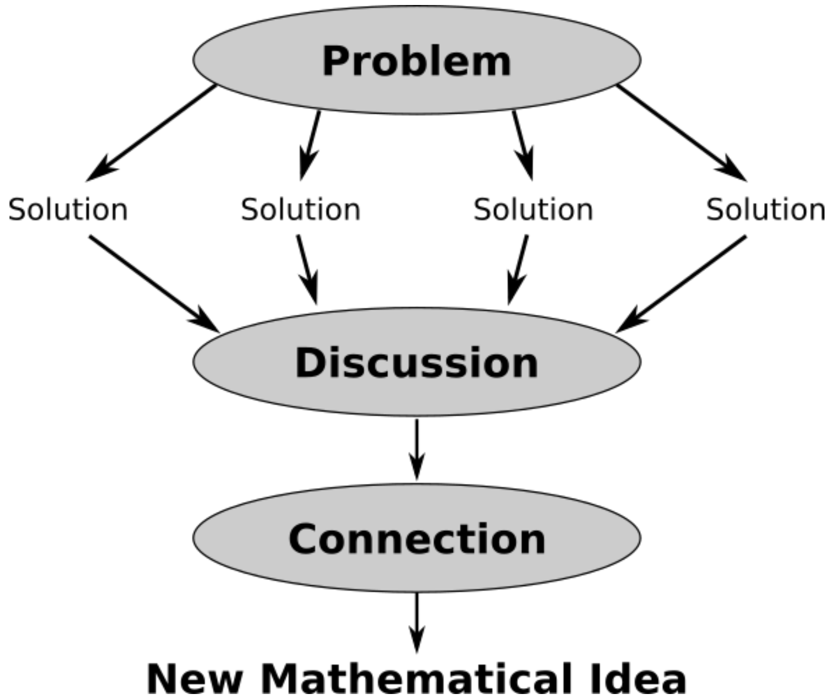

Fig. 1: *The Sequence of Teaching Through Problem Solving.*

The TTPS sequence involves: (1) posing a problem for students to solve, (2) discussing students' solutions strategies, and (3) connecting students' solutions to a new mathematical idea. TTPS is different from teaching problem solving. Teaching problem solving has a different sequence and focus points (see Table 1). It begins with presenting a new mathematical concept and then using this concept in application to solve problems. This approach focuses on understanding procedures for solving problems and acquiring problem solving skills.

Table 1: Teaching Through Problem Solving vs. Teaching Problem Solving

Teaching Through Problem Solving TTPS	Teaching Problem Solving
Helps students learn a concept or procedure through an experience as they engage in problem solving.	Helps students understand concepts first, then using the knowledge gained to solve problems.
Focuses on mathematical discourse and reasoning.	Focuses on learning the heuristics and/or processes of solving a problem.
Allows students to discover multiple ways to solve a problem.	Explains to students ways to solve a problem.

2 An Example

To illustrate the TTPS approach more precisely, the following example taken from a 4th-grade class is provided. Mrs. Hopkins gave her students the following task:

The Field Trip Problem

Eighty fourth-grade students in an elementary school are going on a field trip. Their teachers need to put the students into equal groups to visit the Shark Tank. How many ways can the students be grouped so that each group has the same number of students?

Mrs. Hopkins told her students to find a way to solve the problem and be ready to explain their strategies for solving. Students worked in small groups and were encouraged to speak to one another. The teacher monitored progress and encouraged student-student dialogue while they worked. While monitoring, the teacher decided which solutions she would select for the class discussion and how to sequence the presentation of students' strategies. When the students finished, the teacher asked the pair of students who use a prime factorization approach to present their work followed with another group of students who use a multiplication approach.

Sara and Liza, who chose the prime factoring strategy, described how they found their solutions. Sara said, "We found 5 ways. I divided 80 by two. Then I divided 40 by 2, 20 by 2, 10 by 2, and 5 by 5." Mrs. Hopkins asked, "Would you explain what you mean by that? Why did you start with dividing 80?" After Sara and Liza explained their strategy in detail, Mrs. Hopkins asked students, "Can someone repeat what Sara and Liza said in your own words?" A student said, "They divided the 80 students into two groups to get 40 students in 2 groups. And then, they divided 40 by 2 and got 20 students in 4 groups. Then, they divided 20 by 2 and get 10 students in 8 groups, and so on." Mrs. Hopkins noticed that Sara and Liza did not consider all ways to group the students. For example, Sara and Liza put 40 students in 2 groups but they missed that they could group students differently using the same numbers (2 students in 40 groups). Mrs. Hopkins decided to have another group present and see if this idea would come out in discussion.

Next, Mrs. Hopkins asked Dawn and Jack to describe how they found possible solutions to put 80 students into equal groups. Dawn said, "We found 6 ways; 10 and 8, 40 and 2, 5 and 16." Mrs. Hopkins asked "Could you explain what you mean by that?" Dawn replied, "I mean that the teachers can put 10 students in 8 groups or 8 students in 10 groups. They also can put 40 students in 2 groups or 2 students in 40 groups; and 5 students in 16 groups or 16 students in 5 groups." Mrs. Hopkins restated what Dawn said, "So, you can have 10 and 8 or 8 and 10, you can have 40 and 2 or 2 and 40, and you can have 16 and 5 or 5 and 16. Is that what you said?" Dawn agreed. Mrs. Hopkins turned to Jack and asked, "Do you agree or disagree?" Jack responded, "I don't know. But 80×1 is the same as 1×80." Mrs. Hopkins noticed that Jack had found another way to arrange the students in groups but he was unsure about the use of the commutative operation in the problem context. Thus, Mrs. Hopkins asked, "So, you're saying that 1 student in 80 groups is the same as 80 students in 1 group. Is that what are you saying?" Jack replied, "I thought because $80 \times 1 = 1 \times 80$, this would give us one way of grouping." Mrs. Hopkins then asked the class, "Would someone like to add something more to this?" A student offered, "In this problem, 1×80 and 80×1 are different. We can make 80 groups with 1 student in each or put 80 students in 1 group."

Mrs. Hopkins told the students, "Talk with your partner about how your strategy relates to these two—the prime factorization strategy and multiplication strategy. What did they use or do that was different than what you did?" Mrs. Hopkins wanted to see if Sara and Liza would draw a connection between their solution with three pairs and Dawn and Jack's solution with six pairs. Sara said, "I can now get ten ways to group students because 80/2=40 means 40 students in 2 groups or 2 students in 40 groups, and so on. Mrs. Hopkins asked, "What did your group come up with?" Sara said, "I come to an agreement with Liza that the solutions are all whole numbers that divided 80 into another whole number." The girls made the connection that Mrs. Hopkins hoped for. Sara and Liza wrote down the rest of the factors that they had missed and came to the conclusion that that they were ten solutions.

In this lesson, Mrs. Hopkins incorporated the three phases of the TTPS sequence (see Figure 1). First, she presented the field trip problem to students to solve. Next, she had students share their solutions strategies and why they made sense. Finally, she connected these solutions to the concept of factorization and the important idea that in some situations the order and the labeling of the numbers (number of groups vs number in a group) matters. In this lesson, discussion played a critical role in the development of mathematical ideas.

3 Why is Discussion Significant with TTPS?

Discussion is the central to TTPS. By posing a problem to be solved, students were urged to think of a variety of strategies to solve the problem. When the teacher chooses a problem with various ways to solve and then connects students' strategies and solutions, there is potential for discussion to be mathematically rich. When students' own strategies are used to guide—from what they already know to exploring new mathematical concepts—they learn a concept or procedure through an experience as they engage in problem solving (Stein et al., 2008). This allows students to organize and explore related mathematical concepts.

Engaging students in productive mathematical discussions involves classroom learning environments that encourage and perpetuate meaningful mathematics discussions, engage learners in purposeful inquiry, and lead to a push for conceptual learning (Kazemi & Stipek, 2001). Kazemi and Stipek (2001) describe four norms that help support meaningful mathematical discussions. They refer to these norms as socio-mathematical norms. These norms were evident in Mrs. Hopkins' classroom (see Table 2).

Table 2: Socio-mathematical norms (Kazemi & Stipek, 2001)

Socio-mathematical Norm	Example
Explanations consist of mathematical arguments and not merely procedural descriptions and summaries of the steps to solve the problem.	*Could you explain what you mean by that?*
Errors offer opportunities to reconceptualize a problem and explore contradictions in solutions and alternative strategies.	*So, youre saying that 1 student in 80 groups is the same as 80 students in 1 group. Is that what are you saying?*
Mathematical thinking involves understanding connections among several strategies	*What did they use or do that was different than what you did?*
Collaborative work involves individual accountability and getting consensus through mathematical argumentation.	*What did your group come up with*?

The discussion was focused why solutions made sense mathematically. Mrs. Hopkins looked for opportunities to use students' incomplete thinking as an opportunity. When she purposefully asked Sara and Liza to be the first to offer a strategy, she used their ideas as a starting point to connect them with others' ideas. She did this by asking student to explain their reasoning and evaluate the reasoning of others.

Both student-student and teacher-student dialogue play an important role in high-quality mathematics discussions that are central to a TTPS approach. In order to help students deconstruct their thinking process, it is necessary to allow students to engage in mathematical discussions with their peers. Small group discussion ensures a high level of engagement and promotes students abilities to restate peer thinking in their own words. Students benefit when they discuss their ideas and strategies with other students to formulate their ideas prior to sharing them with the whole class. Then teachers (who use a TTPS approach) can expand these ideas and strategies into whole class discussions.

During whole class discussions, there are several ways teachers can encourage students to learn mathematics through their problem-solving experiences. Chapin, OConnor, and Anderson (2009) outline four important talk moves that promote productive discussions. These four talk moves were represented in Mrs. Hopkinss classroom (see Table 3). By using the talk moves, students were prompted to participate in discussion and synthesize the new mathematics that they were learned. When Mrs. Hopkinss encouraged the students to listen to each other and justify other students reasoning, conceptual and procedural understanding deepened.

Table 3: Talk moves (Chapin et al., 2009)

Talk Move	Example
Revoicing is when the teacher repeats what a student says to make sure that she/he understands what the student intended to say.	*So, you can have 10 and 8 or 8 and 10. Is that what you said?*
Repeating is when the teacher asks students to repeat what another student said; thus, students are encouraged to listen to other students ideas and strategies.	*Can someone repeat what Sara and Liza said in your own words?*
Reasoning is when the teacher asks students to justify other students' strategies or explanations by verifying whether or not they agree with the idea they presented.	*Do you agree or disagree?*
Adding on is when the teacher asks students to create a relation between their own strategies and other students' strategies, or add their own ideas onto already presented ideas of other students.	*Would someone like to add something more to this?*

The discussion in Mrs. Hopkins' lesson was planned. Her development of the lesson drew from Stein et al.'s, (2008) five practices for helping teachers facilitate productive mathematical discussions (see Table 4). Mrs. Hopkins did not randomly have students share solution strategies.

Mrs. Hopkins monitored students' work in small groups to decide which solution strategy should be first, second, etc. and chose the sequence so that it would support the development of the mathematical ideas that were central to the goals of the lesson. The ordering of presented solution strategies, in combination with talk moves and sociomathematical norms, supported the making of connections between solutions.

Table 4: The Five Phases to Orchestrate a Productive Discussion (Stein, et al., 2008)

Phase	Example
1. Anticipating: Involves consideration about how students may interpret a problem, use strategies, and how these interpretations and strategies may relate to the mathematical concepts; and planning to ask effective questions.	Possible solutions of the problem at hand: prime factorization, writing equations, breaking down, and other solutions.
2. Monitoring: Involves making sense of students mathematical thinking and ideas.	Make a table to document what each group did and which representations they used.
3. Selecting: Involves selecting students' responses to share their work with the class.	What solution will be highlighted (the strategy that the teacher wants the students to understand), and who will do it.
4. Sequencing: Involves sequencing students' responses to maximize the chances of achieving the discussion's goals in a coherent manner.	Possible sequence: a. concrete model, b. logical argument, c. algebraic proof.
5. Connecting: Involves drawing connections between students' mathematical ideas and other students' ideas that are presented during the discussion.	Make connections among algorithm solutions, concrete model, and equations that are represented in the groups.

4 Conclusion

Mathematical discussion plays a critical role when using TTPS approach to learn mathematics. During the mathematical discussion in TTPS, students share solution strategies to develop the ability to reason logically and evaluate what makes sense mathematically. This helps students connect and compare their strategies and solutions with the strategies and solutions of their peers. This article described critical teaching practices for facilitating productive mathematical discussions and what they aim to do when using a TTPS approach. These practices involved sociomathematical norms to allow students justify their work, talk moves to make students' thinking visible, and planning discussions in advance to create inquiry-based environment. Facilitating productive mathematical discussions is an essential teaching skill that has a significant impact in the quality of mathematics instructions. This article provided the kind of support needed by mathematics teachers to skillfully implement discussions in TTPS.

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