Modeling with Math Trails

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Abstract: *The authors describe the application of a math trail activity used in a college classroom of future mathematics teachers to model with mathematics. In this activity, pre-service teachers select a location in their community, discover and design five trail heads, model situations using mathematical strategies learned in class, and share in a write-up or presentation. Insights about facilitating a math trail for mathematical modeling are provided, along with suggestions for teachers interested in using math trails in other contexts.*

Keywords: *mathematical modeling, projects, classroom activities, assessment*

1 Introduction

> *Throughout the math trail, I think my group came to the realization . . . that math can be found anywhere. If you're walking to your neighbor's house, or if you're taking a touristic trip in downtown Los Angeles, math can be found anywhere and everywhere.* —Future teacher of mathematics reflecting on a math trail.

Modeling with mathematics plays a large role in solving real-world problems—it allows people to engineer roads, predict cancer growth, and improve weather forecasting. The Common Core State Standards in mathematics (CCSSM) emphasize modeling as an important standard for mathematical practice (SMP) (NGA Center & CCSSO, 2010) (see Table 1). Yet it is not always clear how to teach mathematical modeling to students (e.g., Felton-Koestler, 2016). What practical mathematical activities might engage young mathematicians in modeling with mathematics? In this article, we describe how math trails can be used to engage K-12 students and future teachers, alike, in modeling the world around them.

Table 1: SMP 4—Modeling with Mathematics (NGA Center & CCSSO, 2010)

Modeling with mathematics (SMP 4) involves a student using mathematics to solve a real problem. Engaging in modeling may involve the following activities:

- Pose a question of interest about everyday life, society or the workplace.
- Make assumptions and approximations to simplify a complicated situation.
- Identify important quantities and relationships in real situations.
- Use tools such as diagrams, two-way tables, graphs, flowcharts, and formulas to map relationships.
- Interpret and analyze mathematical relationships within the context to draw conclusions.
- Make predictions that help to answer a question.
- Revise original plans after making sense of initial trials.

In *Guidelines for Assessment and Instruction in Mathematical Modeling Education* (GAIMME), Garfunkel and Montgomery (2016) describe mathematical modeling as "a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena" (p. 8) (see Figure 1). The process includes a number of steps: (a) identifying a problem to better know or understand; (b) making assumptions and identifying variables to obtain an idealized version of the original question; (c) doing math, including computations and problem solving around the idealized question; (d) analyzing and assessing the solution, including whether results from translating back to original problem are practical, reasonable, and acceptable; and, finally, (e) iterating to refine and extend the model.

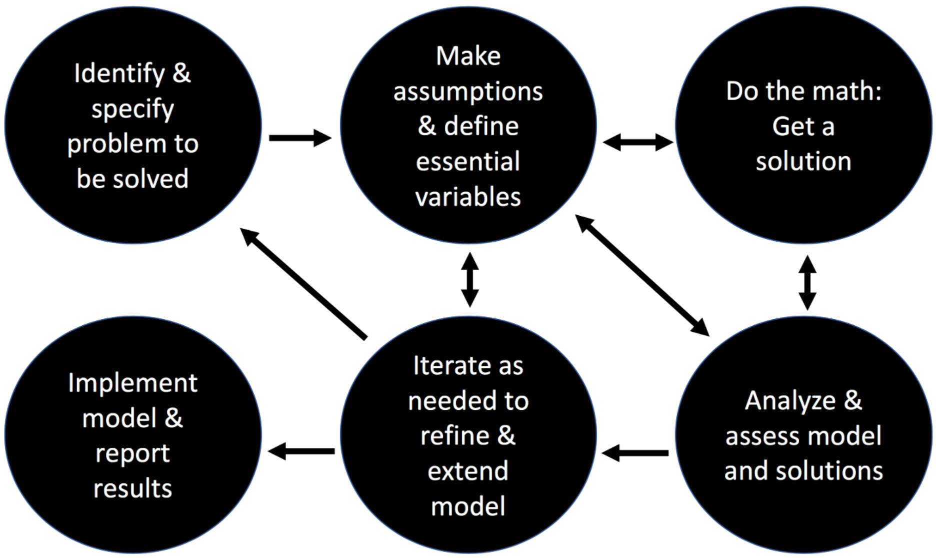

Fig. 1: *Defining mathematical modeling as a back-and-forth process in GAIMME (2016).*

We recognize that modeling takes on different meanings across different contexts. In particular, the word modeling is used in several different ways in mathematics standards. For instance, Felton-Koestler (2016) describes three ways the CCSSM uses model and modeling. One meaning includes representing mathematics using diagrams or physical objects meant to represent mathematical operations or concepts. This includes using concrete models or drawings to add and subtract within 1000 (2.NBT.B.7). A second meaning is as a "stepping stone problem" to "solve relatively straightforward and easy-to-imagine contexts" (Felton-Koestler, 2016, p. 269). An example might include using visual fraction models or equations to solve a word problem (5.NF.B.7). A third view is "using mathematics to solve or understand a messy, ill-defined, real-world problem or phenomenon," which Felton-Koestler refers to as mathematical modeling (p. 269). The math trails we present make use of both mathematical modeling and modeling/representing mathematics.

2 Procedure for a Math Trail

A math trail is an activity that encourages students to explore their environment to discover and notice mathematics around them (Shoaf, Pollak, & Schneider, 2004). Shoaf and her colleagues describe math trails as cooperative instead of competitive, self-directed, voluntary, opportunistic, temporary, and for everyone. Engaging in a math trail involves selecting a physical location and creating problems from the observed environment. Problems may be selected ahead of time by the teacher or discovered during the math trail by students. Skortell (2016) identifies goals of a math trail to include:

- To help students value mathematics by giving them an opportunity to discover its applications in the real world.
- To improve students' critical thinking by giving them an opportunity to create and solve their own problems.
- To improve students' abilities to communicate mathematical ideas.
- To improve students' abilities to collaborate on mathematical tasks.
- To develop students' interest in and respect for the community in which they live.

Teachers can support students in strengthening their mathematical modeling abilities through math trails since the trails encourage students to problem pose questions of interest, make connections among different mathematical ideas, and engage in problem solving. To implement math trails in a college math course for future teachers, we invited students to work in small groups of three to four for a math trail class project. While math trails can be implemented in a variety of ways, preservice teachers in our context were provided with one class to walk around campus to discover five math trails. See Table 2 for directions given to students.

Table 2: Directions for pre-service teachers on how to engage in a math trail.

1. **Assign roles.** Decide who will be the manager, the photographer, note-taker, and/or the creative facilitator.
2. **Pick a specific location.** This could be your campus, a playground, a shopping plaza, a cultural center, a theme park, etc. Your group will visit and search for mathematics in their surroundings. Students are encouraged to think about locations where all group members can meet.
3. **Observe your surroundings.** Notice movement, shapes, structure, quantities, and whatever else that can be described mathematically. Ask questions like, "Can you find a way to estimate the number of tiles in this mosaic?" Aim to find five trail heads, where a trail head is defined as a place on your Math Trail where you stopped and engaged in mathematics.
4. **Identify relevant mathematics content.** Involve the specific material we have discussed in our math class. Challenge yourself to think creatively and to see mathematics hidden in everyday situations. Consider creating your own non-standard unit to measure something (e.g., your shoe length, length of your hand, height). Topics must be connected to ideas discussed in our class (e.g., problem solving, using mental math, fraction reasoning, or integers).
5. **Collect data for each trail head.** Make sure to take detailed notes of your observations and the following information:
    - A motivating question. One question might be: "How many square units are in the pavement if we use someone's foot as a non-standard unit?"
    - A picture of the situation. Take a photograph to communicate the nature of the trail head.
    - A diagram of the situation. This may be a hand-drawn or computer-created diagram. Other diagrams include number lines, double number lines, rectangular area models, etc.
    - Written mathematics symbolizing your situation. Be sure to connect it to the content learned in class.
6. **Summarize.** Work collaboratively to create a presentation and individually to create a summarizing report.

The following math trail was co-created by an undergraduate future teacher and her instructor after the completion of a mathematics content course for future teachers that used math trails. While others have described how math trails can be designed and implemented (e.g., English, Humble, & Barnes, 2010; Richardson, 2004) and provided detailed examples (e.g., Amiya & Kinch, 2014; Shoaf, Pollak, & Schneider, 2004; Skortell, 2016), we suggest how math trails can provide an opportunity to engage students in mathematical modeling. To do so, we highlight two trailheads on our campus. The first example targets the proportional reasoning, which supports a sixth grade math standard on understanding ratio concepts and using them to solve problems (6.RP.A.1-3, 7.RP.A.1-3). The second example targets algebraic reasoning, which supports sixth grade math standards on expressions and equations, particularly on representing and analyzing quantitative relationships between dependent and independent variables (6.EE.C.9, 7.EE.B.4).

3 Planet Walk Trailhead

The Planet Walk is a walkway on campus that contains a scale model of the solar system. Planets are placed in a straight line relative to their actual distance from the sun. Along the approximately 500-foot walkway, a podium is placed in front of each planet, providing information about each (see Figure 2). We were curious to know more about the scale used in the construction of the display. In particular, The Planet Walk motivated us to wonder: *How accurate are the scaled distances of the model compared to actual distances of planets to the sun?*

Fig. 2: *A picture of the first three planets and the sun on The Planet Walk.*

To determine the scale factor of The Planet Walk, we measured each planet's distance from the sun using a non-standard unit of measure called Sarah Units (SU), the length of one of the co-author's feet. We then found the actual distance on the internet and used proportional reasoning to compare the model distances to actual distances. We created a number line diagram relating the number of Sarah Units to each planet's distance to the sun (see Figure 3).

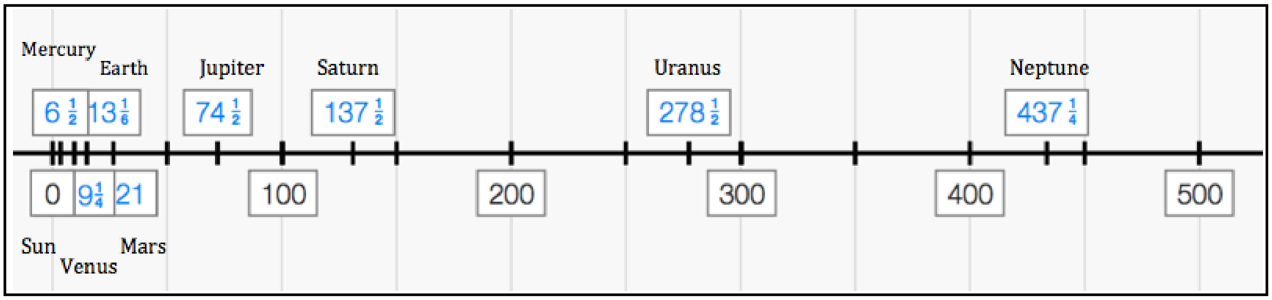

Fig. 3: *Number line diagram for each planet's model distance from the sun in Sarah Units, the length of one of the co-author's feet.*

We found Mercury's distance from the sun to be 6.66 Sarah Units. The actual distance between Mercury and the sun was determined to be approximately 35.98 million miles. We determined a scale factor by dividing the actual distance between Mercury and the sun by the model's distance in SU. We found the actual distance to be approximately 5.4 million times the model distance. This means that each step Sarah took in the model was equivalent to traveling 5.4 million miles in space. We could also say that the ratio 6.66 SU : 35.98 million miles exists, which could be simplified to 1 SU : 5.4 million miles.

We continued the process of finding ratios of each pair of model and actual distances to test for equivalent ratios (7.RP.A.2.A) (see Table 3). Sometimes we swapped who measured the distances, as we noticed that both co-authors' shoe sizes were approximately equal in length. We made additional approximations by estimating distances from previous measurements (rather than starting anew each time from the sun) to make the data collection process more efficient.

Table 3: Planet and model distances.

		Trial 1		Trial 2	
Planet	**Distance from Sun (millions of miles)**	**Model Distance from Sun (Sarah Units)**	**Scale Factor**	**Model Distance from Sun (Sarah Units)**	**Scale Factor**
Mercury	35.98	6 2/3	5,397,053.97	6 1/2	5,535,384.62
Venus	67.24	10 1/2	6,403,809.52	9 1/4	7,269,189.19
Earth	92.96	15 1/2	5,997,419.35	13 1/6	7,060,074.43
Mars	141.6	23	6,156,521.74	21	6,742,857.14
Jupiter	483.8	75	6,477,333.33	74 1/2	6,520,805.37
Saturn	888.2	135 1/3	6,563,055.80	137 1/2	6,459,636.36
Uranus	1,787	276 2/3	6,459,037.70	278 1/2	6,416,517.06
Neptune	2,795	434	6,440,092.17	437 1/4	6,392,224.13

After determining the scale factors for model/actual distances to the sun, we wondered if each pair of distances shared the same constant of proportionality (7.RP.A.2.B). We discovered that not all pairs shared the same ratio. Scales varied from 5.3 million miles/1 SU to 6.5 million miles/1 SU. We reflected on why the scale factors varied if the Planet Walkway was likely designed using the same single scale factor. Doing so provided us an opportunity to analyze and assess our data, and consequently refine our process.

We decided to recollect our distance data to ensure greater precision and to strengthen our conclusions. During Trial 2, we systematically marked benchmark numbers with chalk to keep track of distances and measured lengths in as straight of a line as possible. We also ensured that each SU was measured heel-to-toe in an effort to minimize measurement error and decided to use only Sarah's foot length for consistency. Finally, we decided to measure from the location of the model sun to the front of the approximately 3-inch long planet posts, as opposed to its center or far end. We hypothesized that these revisions would provide more accurate data from which to draw conclusions.

Table 4 displays how scale factors compared in each trial. Notice that the average scale factor for all combinations of sun-to-planet measurements was lower in Trial 1 than in Trial 2. The scale factors had a range of 1,200,000 in Trial 1 and 1,700,000 in Trial 2. This was interesting as we hypothesized that the range would be smaller with more accurate data collection techniques. The medians differed by an additional 25,000 miles for Trial 2. Reasons for variation include our imprecise measuring method, using multiple non-standard units (both Sarah and Bridget Units), the issue of consistently measuring from the end point of each length and its post, and counting errors. Alternatively, variation may also be explained by the design of the Planet Walk. We concluded that there may have been some inconsistency with the scale factor used to build the model system, but recognized our own imperfections in the mathematical modeling process.

Table 4: Scale factor statistics across two trials.

	Trial 1	**Trial 2**
Average scale factor	6,236,790.4475	6,549,586.0375
Range of scale factor	5,300,000 – 6,500,000	5,500,000 – 7,200,000
Median scale factor	6,460,000	6,485,000

3.1 Mathematical Modeling

Reflecting on our process, we engaged in mathematical modeling using proportional reasoning in several ways:

- We identified a problem regarding the accuracy and proportionality of our solar system represented on the sidewalk.
- We defined a non-standard unit of measure, the Sarah Unit, to measure distances and made measurement approximations along the way.
- We used tools, such as a number line diagram, tables organizing length measurements and averages, ranges, and median, and calculators to represent mathematical relationships between collected data points.
- We analyzed and assessed our model, questioning whether our scale factors made sense.
- We iterated our process twice, refining our data to better answer our problem.

Although modeling a scaled version of the solar system was specific to a particular location at our school site, proportional thinking can be discussed in other situations. Students could compare rates of climbing sets of stairs, walking speeds between several locations, and the steepness of different architectural structures.

4 Brick Border Design Trailhead

Walking around the art buildings on campus with sidewalk chalk, we noticed bricks on the ground and wondered about growing patterns learned in class. We began to count the number of bricks in the border of a square (see Figure 4). We imagined that it would be useful to collect data to find a formula to predict how many bricks would be necessary in any sized square border (activity inspired by Boaler, Humphreys & Ball, 2005).

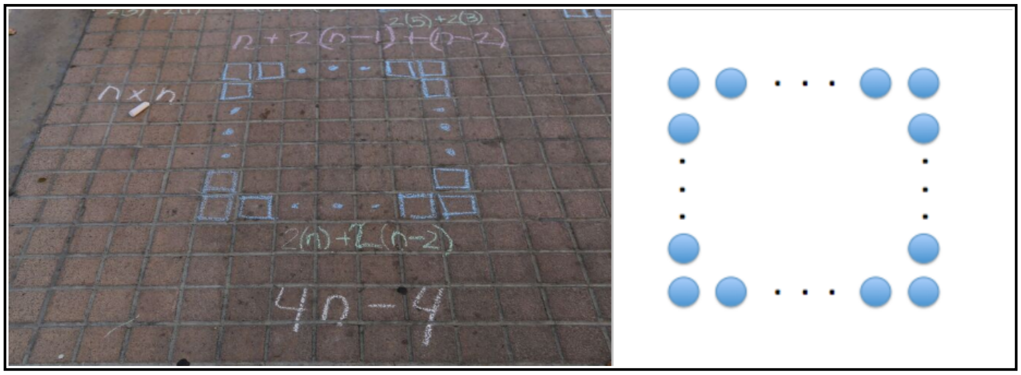

Fig. 4: *The nth case of an $n \times n$ border task.*

To find a way to count the number of bricks needed for a general border without counting one by one, we decided to find the number of bricks in specific and simple cases: 3×3, 4×4, 5×5, 6×6, and 10×10 borders. We discovered three different expressions that count the number of bricks given the size of the square, n, where n represented the length of one side (6.EE.B.9). Table 5 provides the three different expressions.

Table 5: Three different expressions for number of bricks in an $n \times n$ border problem.

Dimension	**Expression 1:** $n + 2(n-1) + (n-2)$	**Expression 2:** $2(n) + 2(n-2)$	**Expression 3:** $4n - 4$
3×3	$3 + 2(3-1) + (3-2) = 3 + 2(2) + 1 = 8$	$2(3) + 2(3-2) = 2(3) + 2(1) = 8$	$4(3) - 4 = 8$
4×4	$4 + 2(4-1) + (4-2) = 4 + 2(3) + 2 = 12$	$2(4) + 2(4-2) = 2(4) + 2(2) = 12$	$4(4) - 4 = 12$
5×5	$5 + 2(5-1) + (5-2) = 5 + 2(4) + 3 = 16$	$2(5) + 2(5-2) = 2(5) + 2(3) = 16$	$4(5) - 4 = 16$
6×6	$6 + 2(6-1) + (6-2) = 6 + 2(5) + 4 = 20$	$2(6) + 2(6-2) = 2(6) + 2(4) = 20$	$4(6) - 4 = 20$
10×10	$10 + 2(10-1) + (10-2) = 10 + 2(9) + 8 = 36$	$2(10) + 2(10-2) = 2(10) + 2(8) = 36$	$4(10) - 4 = 36$
$n \times n$	$n + 2(n-1) + (n-2)$	$2(n) + 2(n-2)$	$4n - 4$

The first expression we found was $n + 2(n-1) + (n-2)$ (see Figure 5). We counted all bricks on the left side of the border (n), the amount of bricks on the top and right sides of the border without counting the bricks that we already counted (2 groups of $n-1$), and the number of bricks that remained on the bottom of the border ($n-2$). We used variables to represent mathematics and combined like terms to give us $n + 2(n-1) + (n-2)$. While this simplifies to $4n - 4$, we left it in the form $n + 2(n-1) + (n-2)$ since this expression corresponded to the way we structured our diagram.

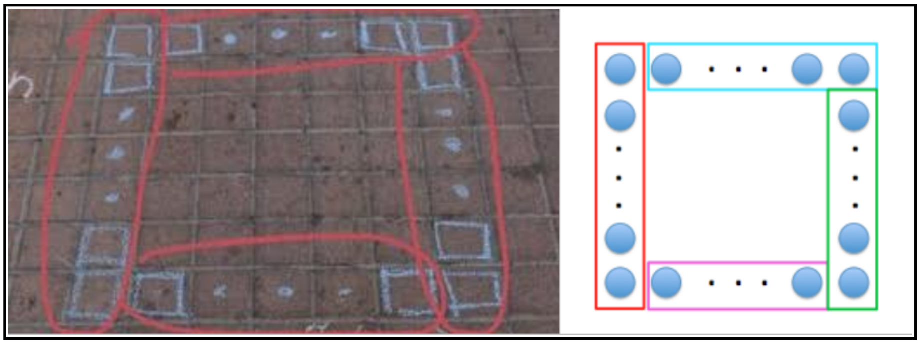

Fig. 5: *The nth figure of an $n \times n$ border represented by the expression $n + 2(n-1) + (n-2)$.*

Another expression we discovered was $2(n) + 2(n-2)$ (see Figure 6). To make sense of this expression, we first counted the amount of bricks on the top and bottom rows of the border. This gave us $2n$, since both sections of bricks had n bricks. Next, we counted the number of vertical bricks on the left and right side of the border without double-counting any squares. This gave us $(n-2)$ bricks for both the left and right border, or $2(n-2)$. We subtracted two from n because we had already counted two bricks on the right and left side while counting the number of bricks in the top and bottom rows of the border. Adding these terms gave us $2(n) + 2(n-2)$, which also simplified to $4n - 4$.

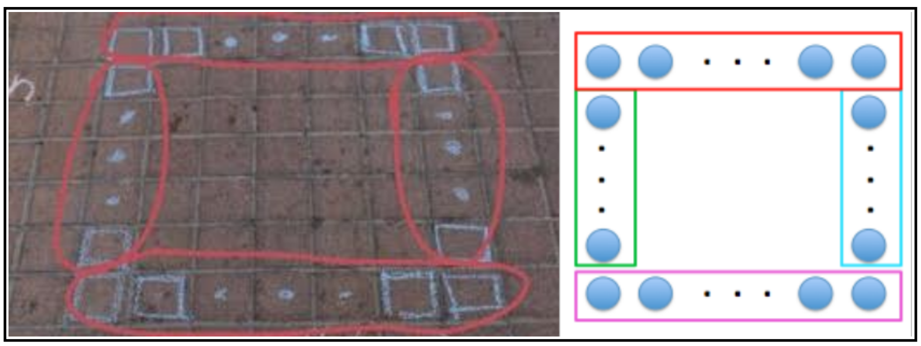

Fig. 6: *The nth figure of an $n \times n$ border represented by the expression $2n + 2(n-2)$.*

For the last expression $4n - 4$, we counted the amount of bricks on all the sides, which gave us us $4n$ since there are 4 sides with n bricks in each side (see Figure 7). We then subtracted four bricks from our expression because of the amount of bricks that overlapped in each corner.

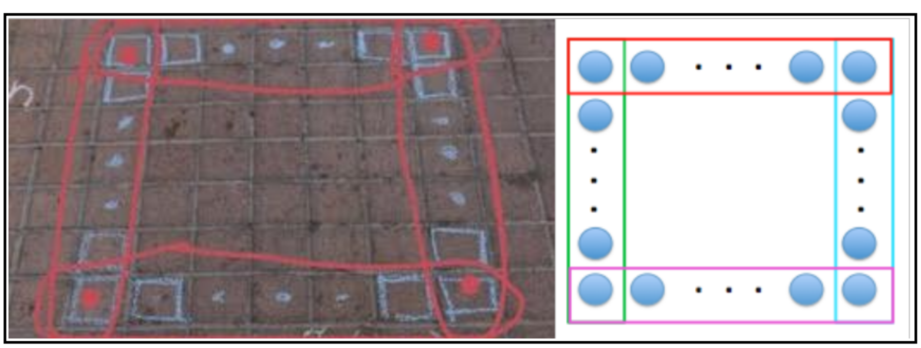

Fig. 7: *The nth figure of an $n \times n$ border represented by the expression $4n - 4$.*

4.1 Modeling with mathematics

We modeled with mathematics while engaging in algebraic reasoning in several ways:

- We used diagrams to represent the total number of bricks given any $n \times n$ border, and related the set of abstract symbols to the quantities in the diagram.
- We described several expressions that could represent the way that the pattern grew and connected each expression to a diagram: $n + 2(n-1) + (n-2)$, $2(n) + 2(n-2)$, and $4n - 4$.
- We answered a real question as to how many bricks or tiles would we need to design any border, useful for designing and constructing patterns with tiles.

5 Using and Extending Math Trails

Engaging students in a math trail within the elementary, middle, or high school setting is a feasible activity that could serve as a vehicle for learning mathematics topics and modeling. As the GAIMME report states, "mathematical modeling should be taught at every stage of a student's mathematical education" in part because "mathematics is important in dealing with the rest of the world" (GAIMME, 2016, p. 7). Teachers could find ways to use math trails as an introductory activity, formative in-class assessment, or summative project. For an introductory activity, teachers may invite students to measure shadows around campus to introduce a discussion of ratios. Students might be encouraged to estimate the height or weight of objects to begin conversations around estimation.

As a formative assessment, teachers may pre-select an area for students to observe and design a sequence of questions related to a topic. In the Watts Tower Math Trail (California Mathematics Network Region XI Los Angeles County, 2016), one location focused on a building design involving a mosaic of glass bottle bottoms arranged in a roughly semi-circular pattern. Several questions were listed that could be asked to assess what students noticed about the math trail:

- An elementary question asked, "About how many circles do you see here? What makes them circles?" (p. 7).
- A middle school question read, "Invent a method for counting the circles" and "Use your method to count the circles." (p. 7).
- A high school question stated, "At the right is the same picture as above, with a coordinate system placed on it. Add scale, and determine an equation for the parabola formed" (p. 7).

As a summative assessment, students may be invited to product a report or presentation about their unique math trail. In all situations, students have the opportunity to creatively connect mathematics learned in class to real situations of interest in their communities. Whether students model using mathematics or engage in the back-and-forth process of mathematical modeling, math trails provide an opportunity for doing rich mathematics.

With curiosity, planning, and exploration, teachers can invite students to apply a mathematical lens to the world around them. Math trails are adaptable activities that provide opportunities to see mathematics as a tool for making sense of the world, to communicate and collaborate mathematically, to deepen understandings of the purpose of previously isolated mathematical topics, and to connect students to their community.

6 Concluding Remarks

Math trails create opportunities for students to connect mathematics content learned in a classroom to the real world. This process involves both mathematical modeling as seen with the Planet Walk example and modeling with mathematics as seen in the Brick Border Design example. One student who engaged in Math Trails as part of their future teacher math course reflected,

> This assignment helps you to think more abstractly. When we are teaching fractions or something else to a class, we can use this experience and real life examples to help them [students] relate to math more.

Another student reflected, "I have to admit that at first it wasn't very easy because everywhere you look there is mathematics involved." The notion that mathematics could be found everywhere was shared by many students who engage in a math trail. One student, who played softball for her university, noted that,

> I would have never thought to look at the trailheads that I did before this class. I do not know if anyone has ever used any of the sport facilities on campus for a math trail, but I find it very interesting how much math surrounds these areas.

The details of math trails presented here provide the reader with examples of engaging in modeling around proportional reasoning and algebraic thinking. In the course of our math trail, we posed and answered our own questions, made connections between mathematics and the real world, used tools such as diagrams and proportional and algebraic thinking to problem solve, updated and revised our methods of collecting data to improve accuracy, and collaborated to clearly communicate our thinking. The activities of actualizing the size of our solar system and using models to design patterns engage students in doing mathematics from the lens of modeling. For students, math trails have the potential to shape the extent to which mathematics can be seen and created in their environment. For teachers, engaging in math trails provides an informal assessment opportunity to gauge levels of mathematical modeling and their understanding of mathematical topics, all while connecting what is learned within a classroom to the world outside of it. Opportunities provided through math trails can help to connect students and teachers to their local surroundings and, in doing so, support the development of rich and meaningful mathematical modeling abilities.

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