Shapes and Their Equations: Experimentation with Desmos

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Abstract: *The author explores common polygons—squares, triangles, trapezoids—and uses the free online grapher, Desmos, to derive single equations and inequalities that generate the shapes. Throughout, the author describes his inquiry-based approaches—the trials and errors that he employs in his eventual discovery of various shape-generating expressions.* ***Keywords:*** *polygons, graphing, inquiry, multiple representations*

1 Introduction

Mathematics has always interested me. As a student at Columbus Academy, I have the opportunity to study with incredible teachers who spark new interests and help me find answers to mathematical questions. Lately, I have been researching different shapes and their equations. For instance, the equation, $x^2 + y^2 = 1$, generates a circle centered at the origin with radius 1 unit. While my textbook provides equations for conic sections—ellipses (including circles), hyperbolas, and parabolas—the authors provide no single equation to generate other simple shapes (e.g., squares and triangles). Ever since I discovered this, I have been on a quest to find different ways to express common shapes with single equations. Using Desmos, a free online graphing utility, I've worked to develop, test, and revise conjectures and share findings with teachers and classmates.

2 The Search For A Square

My first idea was the square. The more common way I was taught to make a square was by creating lines at different x and y values. The results would look somewhat like Figure 1.

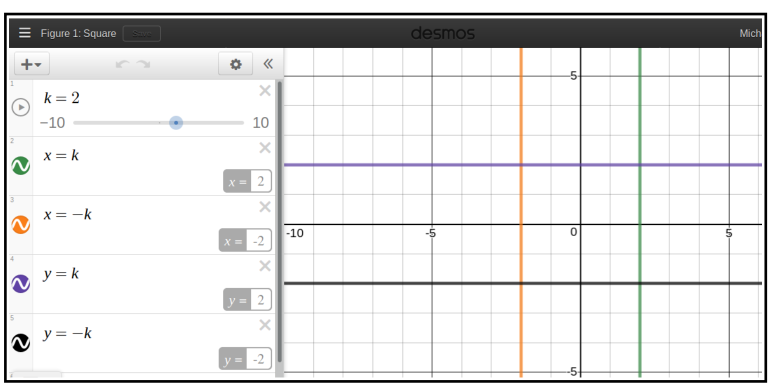

Fig. 1: *Graphing a square with four perpendicular lines (see https://tinyurl.com/squarewithlines)*

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This was okay, but I just wanted the square. I didn't want lines that went on forever in different directions. Of course we could also restrict the domain or range of these lines, such as typing $y = 1\{-1 \leq x \leq 1\}$, to restrict the graph to a square, but this was not satisfying. I started playing with the circle equation. I found that if you were to make the powers of x and y larger while keeping them equal to each other, a square-like shape would be created. I got all the way to $x^{512} + y^{512} = 1$ before I realized that it would never create a perfect square, because at each angle of the square it would still be a curve, as shown in Figures 2 and 3. To experiment further with this, I used a slider in Desmos. To explore my work, navigate to the following link: https://tinyurl.com/almostsquare.

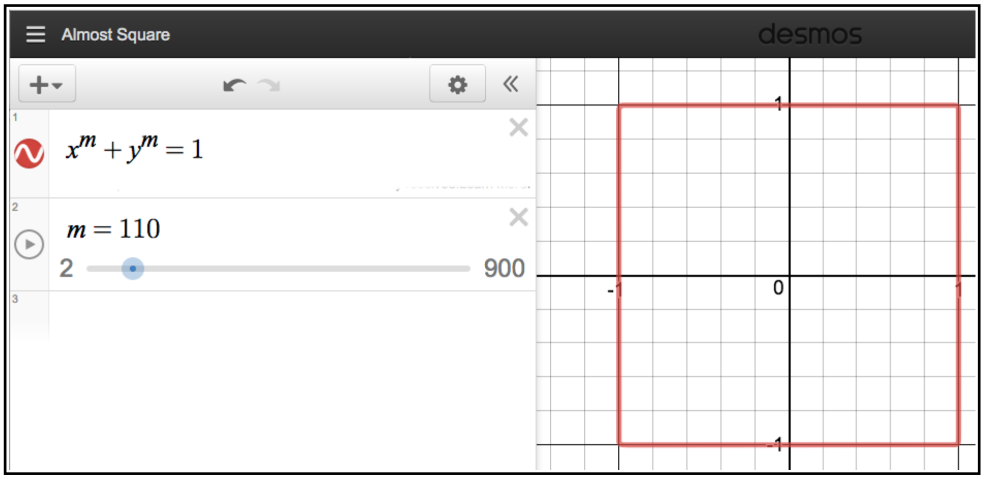

Fig. 2: *Graph of $x^{110} + y^{110} = 1$, a square-like shape (see https://tinyurl.com/almostsquare).*

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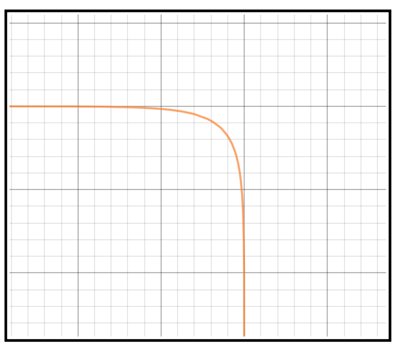

Fig. 3: *Local behavior at one "vertex" of the square-like shape*

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Then one of my teachers suggested that there is an equation for a square. This equation was: $|x| + |y| = 1$. This created something that resembled a rotated square. It technically was a square; it had four sides that were equal in length. However, it was rotated about the origin as shown in Figure 4. The main goal was to find an equation that would create a non-rotated square. Soon, I was able to find one using many different equations. I wanted to use infinite values for m and n, which didn't work since Desmos did not allow such input. Then, I tried to find an equation for segments so I could construct a square, but then I realized that this wouldn't be a single equation.

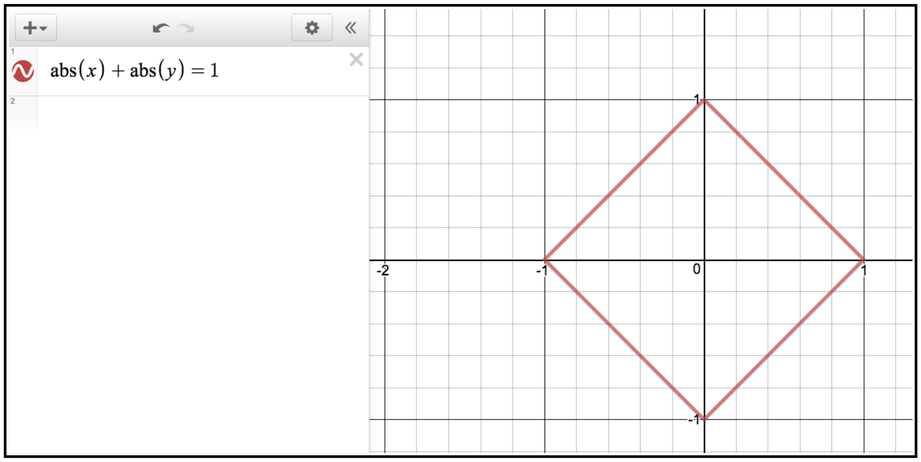

Fig. 4: *Graph of a rotated square generated by equation* $|x| + |y| = 1$ *(see https://tinyurl.com/rotatedsquare).*

Finally it occurred to me that an equation that generates a square actually generates a set of points. This observation led me to the equation $max(abs(x), abs(y)) = 1$. It's instructive to look at a table of values for sample x and y to better understand the behavior of the expression.

x	y	abs(x)	abs(y)	max(abs(x),abs(y))	max(abs(x),abs(y))=1?
0	0	0	0	0	no
0	0.5	0	0.5	0.5	no
1	0	1	0	1	yes
1	0.1	1	0	1	yes
1	0.2	1	0	1	yes
1	0.9	1	0	1	yes
1	1	1	1	1	yes
1	1.1	1	1.1	1.1	no

As the table suggests, all points $(1, y)$ with $0 < y < 1$ satisfy $max(abs(x), abs(y)) = 1$. I used the max function to ensure the use of every point in the coordinate plane. The absolute value function ensures that (x, y) are plotted for negative x and y. The resulting graph is shown in Figure 5.

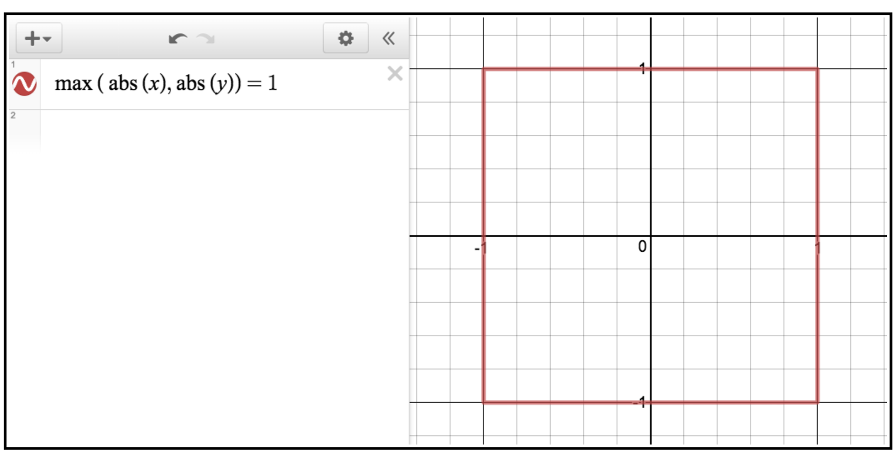

Fig. 5: *Square generated by* $max(abs(x), abs(y)) = 1$ *(see https://tinyurl.com/maxfunctionsquare).*

3 Explorations With Other Shapes

The previous equation, $max(abs(x), abs(y)) = 1$, can be altered to produce different shapes. Specifically, we can produce rectangles, triangles, and trapezoids of all different sizes. We can use the square equation from Figure 5 as a starting point for other findings. Altering this equation leads to the discovery of shapes such as triangles, trapezoids, and rectangles. The equations of these shapes have a similar format to that of the square. Let's start by considering a triangle.

3.1 Triangle

The triangle-generating expression, $max(abs(-2y), abs(4x) + 2y) \leq 1$, is shown in Figure 6. Note that the constant term on the right appears to represent the height of the triangle—if we set the right side of the inequality to 1, the triangle's distance from the origin is 1. *Does that property hold for any value of the parameter? Drag on the slider and explore!* The factor multiplied by x controls the width of the triangle. Smaller values appear to make the triangle wider. Note, too, that the triangle appears isosceles—the vertices on either side appear to be equidistant from the y-axis.

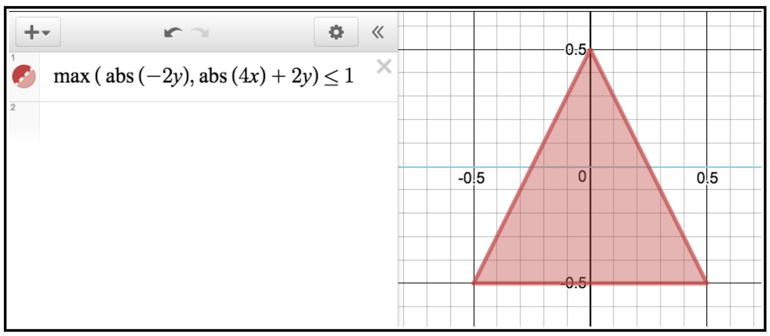

Fig. 6: *Graph of triangle and its interior (see https://tinyurl.com/triangleequation).*

3.2 Rectangle

The generating expression for a rectangle is similar to that of the square. For example, consider the inequality $max(abs(2y), abs(4x) - 2^2) \leq 2$. The values that multiply by x and y give the rectangle its height and width.

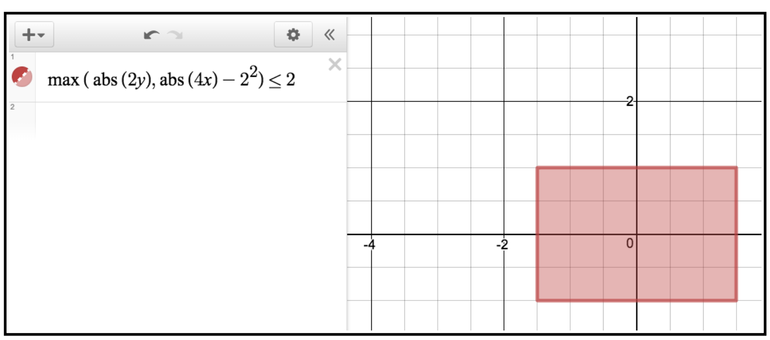

Fig. 7: *The graph of a rectangle and its interior (see https://tinyurl.com/rectangleequation).*

Surprisingly, setting the x and y coefficients equal does *not* generate a square. Compare the results of the following expressions in Desmos. Which generates a square? (a) $\max(\text{abs}(2y), \text{abs}(2x) - 2^2) \leq 2$; (b) $\max(\text{abs}(2y), \text{abs}(6x) - 2^2) \leq 2$

3.3 Trapezoid

Since a trapezoid can be viewed as a truncated triangle, the generating expression for the two shapes are closely related. Consider, for instance, the expression $max(abs(5y), abs(4x) + 2y) \leq 2$. Note that the coefficient of y in the absolute value appears to be larger than the constant value on the other side. *Does this property hold for any trapezoids?* The link https://tinyurl.com/trapezoidgraph contains a more generalized graph that can be manipulated to explore this conjecture (and many others) in greater depth.

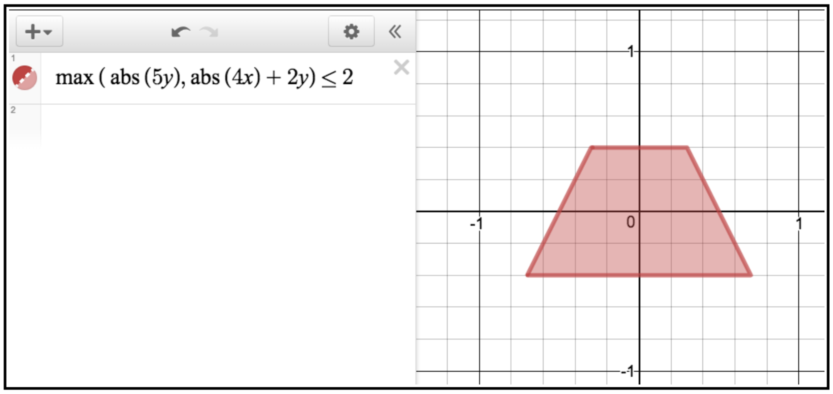

Fig. 8: *The graph of an isosceles trapezoid and its interior (see https://tinyurl.com/trapezoidequation).*

You can make many new shapes using the expressions I've provided by changing numerical values of parameters. The beauty of these expressions is their versatility—all are readily manipulated using parameters and sliders in Desmos.

4 Summary

This project has provided me with answers to questions that I didn't even know I had. My hope is that the expressions I've provided can be used to spark further mathematical explorations. The use of Desmos and other electronic tools that create a more dynamic approach to mathematics proved invaluable to me in this project.

I would like to thank Chris Bolognese, my math teacher, who helped me fine tune this project and give me pointers and ideas throughout my research. Mathematics is not just lines on graphs, but it is the movement and manipulation of those lines on graphs that make it so interesting. I hope more students will ask and explore their own questions, developing their own methods for investigating problems of their own design. Such work benefits the mathematical knowledge of the researcher, but also impacts the knowledge of student peers and teachers. I firmly believe that teachers should encourage their students to pursue their own projects and ideas that will broaden their knowledge of mathematics. I'm positive that other students like myself can research concepts such as these. I will certainly continue my exploration of mathematics and I hope to uncover more interesting mathematics with Desmos.

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