Survival of the Fitness: Mathematical Activity as a Union of Content and Process

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Abstract: *Guided by an unexpected trigonometric argument offered by a student, the author proposes an evolutionary metaphor of fittest versus fitness as a way to envision mathematical activity that honors the intent of standards documents that include both process and content standards.*

Keywords: *trigonometry, law of sines, standards, mathematical practices*

1 Introduction

Modern standards documents in mathematics education contain two categories of learning standards: content and process standards. This holds true for the Ohio Department of Education's renewed learning standards (2017), with the list of content standards preceded by eight Standards for Mathematical Practice (SMPs) taken from the Common Core (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010). It is simple to propose that classroom instruction enfolds all content standards in the SMPs, but their union is complicated by fundamental differences in their character. For instance, the SMPs are not attached to any specific domain of mathematics; rather, they transcend and connect content boundaries. Also, the SMPs are not ranked and do not exist on a developmental timeline. Although they are presented as an ordered list, their numbers are labels, not rankings. In contrast, the content standards sort each conceptual category (e.g., geometry) into a series of domains, each containing clusters of standards ordered in terms of difficulty, ending with advanced standards exempt from state testing. As opposed to process standards, content standards attempt to cordon off sub-sections of content and present them as hierarchies to be mastered sequentially.

2 Optimization and Fittest v. Viability and Fitness

To further articulate differences between content and process standards, I use the evolutionary terms *fittest* and *fitness*. The progression of content standards implies that there exists an optimal—a fittest—way to approach specific mathematical problems. Content is covered by incrementing difficulty then providing optimal methods for resolving the new problems. Each of these fittest methods becomes the "industry standard" until a new wrinkle is introduced that triggers learners to develop more refined approaches. In each case, the teacher supports the development of new procedures, alleviates pressure by guiding students to a new fittest method, and continues the march through content standards. On the other hand, process standards imply that mathematical knowing is about viability—a fitness—of a knower in a mathematical environment. A learner maintains mathematical fitness by considering the possibilities a problem offers and acting in a manner that preserves the potential for further actions. In short, an action contains fitness if it allows the learner to make mathematical sense of the given situation. This view of mathematical activity makes room for students to structure mathematical work, make mathematical decisions, construct mathematical sense, and justify that sense-making activity in a variety of ways. Therefore, the environment does not force specific mathematical actions, but it affords certain possibilities; doing mathematics becomes a process of leveraging those possibilities. In other words, the spirit of the process standards is not about diagnosing and executing the next fittest procedure in a series; rather, they are about considering the fitness of a plethora of mathematical possibilities while moving toward the resolution of a problem.

The reality is that teachers habitually rely on content standards in their planning, which means that courses are typically driven by the progression of content outcomes. While process standards may still play an important role in daily activities, teachers' focus on the progression of content inadvertently communicates to students that learning mathematics is only about mastering new, more sophisticated methods for solving new, more sophisticated problems. To be clear, this progression of content is not an issue in and of itself. Rather, the concern is the illusion that any new methods become the sole option for resolving new problem scenarios. It is too easy for teachers and students, alike, to obsess about following content standards and to lose touch with the connective, recursive vision of the SMPs: that the process of doing mathematics involves a learner applying content knowledge to make sense of new problem solving situations. In this article, I present an image of mathematical fitness where a student's trigonometric argument jolted me from my progression toward a fittest method and provided me with an image of the process standards in action.

3 My Classroom

Our trigonometry unit on the Law of Sines and Cosines began with the introduction of an inconvenient triangle (Figure 1). The class quickly determined that employing the primary trigonometric ratios no longer made sense in the absence of a right angle. Luckily, I possessed (and was willing to share) a new, fittest way to the new, more complex problem of solving a non-right triangle. Actually, I didn't have much choice in the matter; my content standards mandated that I do so.

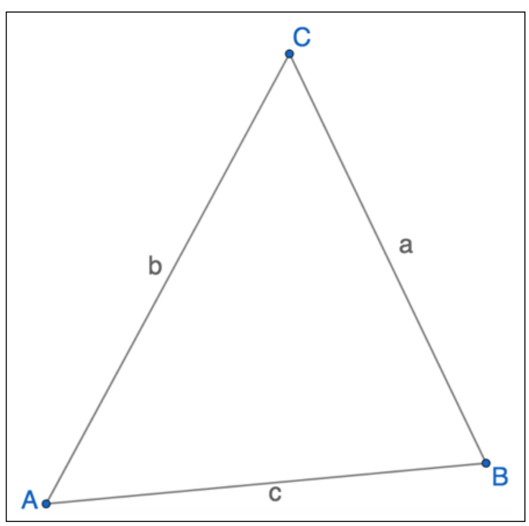

Fig. 1: *An inconvenient triangle.*

3.1 "Proof" of the Fittest

I began with a proof of the Law of Sines that relies on the construction of a pair of right angles in a non-right triangle. I reproduce it here along with a brief commentary provided to my students. The proof begins by dropping an altitude in a generic acute triangle (Figure 2).

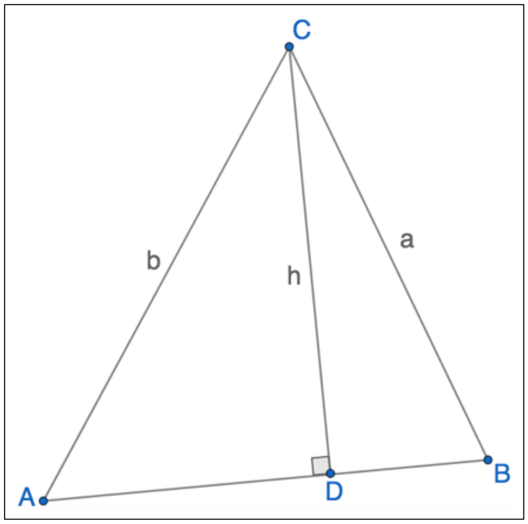

Fig. 2: *$\triangle ABC$ with altitude h intersecting $\overline{AB}$ at point D.*

We recover, for the time being, the right angle necessary to work with primary trigonometry by dividing the inconvenient triangle in two. I was quick to express the annoyance of this necessity. I want to push trigonometry forward where it can tackle triangular problems without having to worry about right angles. In order to achieve this goal, I began two simultaneous lines of argument:

$$\begin{aligned} sin(A) =& \frac{\text{Opposite side}}{\text{hypotenuse}} \\ sin(A) =& \frac{h}{b} \\ b \cdot sin(A) =& h \end{aligned} \qquad\qquad \begin{aligned} sin(B) =& \frac{\text{Opposite side}}{\text{hypotenuse}} \\ sin(B) =& \frac{h}{a} \\ a \cdot sin(B) =& h \end{aligned}$$

Since the length of the altitude, h, is equal to both expressions involving the sides and angles of the original triangle, the two expressions are also equal. This transitivity connects the two arguments, resulting in the Law of Sines.

$$\begin{aligned} b \cdot sin(A) =& h = a \cdot sin(B) \\ b \cdot sin(A) =& a \cdot sin(B) \\ \left(\frac{1}{a}\right)\left(\frac{1}{b}\right) b \cdot sin(A) =& a \cdot sin(B) \left(\frac{1}{a}\right)\left(\frac{1}{b}\right) \\ \left(\frac{1}{a}\right) sin(A) =& sin(B) \left(\frac{1}{b}\right) \\ \frac{sin(A)}{a} =& \frac{sin(B)}{b} \end{aligned}$$

At this point, we returned to the diagram (Figure 2) and erased the altitude to leave the original triangle (Figure 1) void of any trace of right angles. With that action, we officially installed a new industry standard for triangle trigonometry, one that allowed us to retire the methods of right-angled triangles in favour of a more sophisticated alternative.

3.2 The task

For homework, I asked my students to find the unknown measurements in an acute triangle with the intention of beginning the next class with a discussion of possible struggles (see Figure 3). The intent of the task was to help me better understand my students' initial understandings of the Law of Sines.

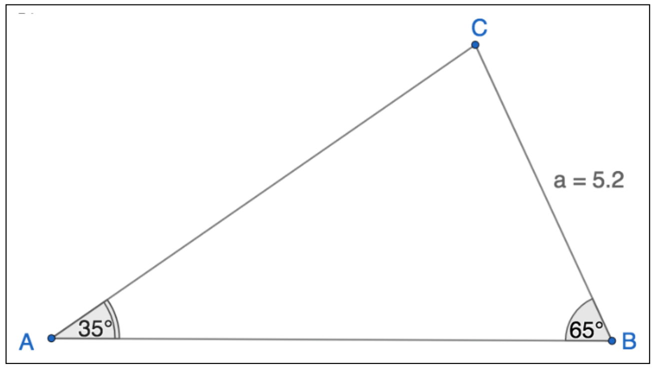

Fig. 3: *A homework problem for the Law of Sines.*

4 An Image of Fitness

At the beginning of the next class, it appeared that my students had made the jump of sophistication successfully and were able to use trigonometry in a triangle lacking a right angle. I was ready to provide a new challenge when Tyler provided an unsolicited suggestion:

TYLER You don't need to do the Sine Law at all.

ME How so? You're stuck using the Sine Law because you don't have a right angle.

TYLER What if we make a right angle?

In another unsolicited move, Tyler made his way to the board, sketched the altitude onto the homework problem (Figure 4), and argued that if you can get two pieces of information in any right triangle, you can solve the triangle.

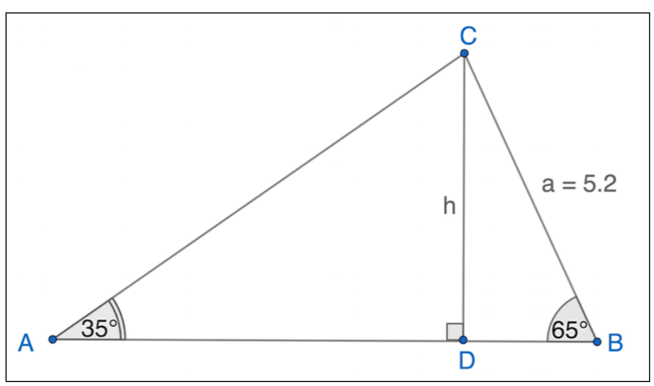

Fig. 4: *Tyler's altitude, h, means that the Sine Law is no longer necessary.*

TYLER We already have two pieces in $\triangle CBD$, so we can find the three missing pieces. We don't even need the angle, because we can sum to 180, so that's only two calculations. Once we do that, we have two pieces in $\triangle ACD$ and can then solve for the last two missing pieces.

He did not even bother to carry out the calculations to verify that his argument resulted in identical results. Rather, his argument was based on our proof from the previous class meeting, relying on the structure of triangles and the utility of trigonometry. Ironically, the proof that was intended to establish the Sine Law as the fittest way to handle problems like these ended up providing an alternative avenue of fitness to arrive at a solution.

TYLER So, we never need the Sine Law, but it might be useful in the future—I don't know.

ME But isn't the Sine Law more efficient?

TYLER How? You need to do the Sine Law twice to get the solution, and I need to do four smaller calculations. I think my way is more efficient.

It was clear that Tyler had considered the Sine Law as an option in completing the homework problem, and he thought that it might be a relevant method for solving future problems. However, his actions indicated that he approached the problem with a different lens. While I was caught up in presenting the next fittest procedure, implicitly branding the triangle a "Sine Law problem," Tyler recognized the homework problem as a task for which the Sine Law was one of many solution possibilities. Rather than dispose of the previous artefacts of trigonometry, he made an informed decision about which he would employ to construct his solution. In doing so, he stepped out of line with the march toward the next method, and made sense of the problem (SMP1), constructed a viable argument (SMP3), used appropriate tools strategically (SMP5), made use of the geometric structure (SMP7), and expressed repeated reasoning (SMP8) by extending his argument to the conclusion that the Sine Law might be considered a trigonometric luxury rather than a necessity. He was not enamored with the fittest method. He was interested in the argument that made sense as he interacted with the problem, and the content standards lost their sense of optimization as he weighed their affordances. In other words, he was acting to maintain his mathematical fitness.

5 Implications for Teaching

My intention here is not to claim that the inclusion of both process and content outcomes in modern curricula documents is nonsensical, nor that one type of standard is more important than the other. In fact, creating some sort of hierarchy of standards—a fittest—would be exactly opposed to the intended message. Rather, it is to claim that teachers habitually allow the sequential mastery of content standards alone to determine the mathematical proficiency of learners. In others words, we default into a mode that mimics the survival of the fittest. Content is intended to be encountered through the process standards, and the process standards are enacted through the vehicle of content. The two coalesce when students act mathematically to interrogate the suitability of content in specific scenarios.

Encouraging and influencing this fitness of mathematical knowing requires a shift in how we observe both mathematical content and students' mathematical activity. It requires mathematics content to be respected for its interconnected nature. That is not to say that an increase in complexity is not important, but complexification need not imply a mathematical amnesia for things past where the introduction of new methods makes previous ones obsolete. For instance, I am not saying that the Sine Law has no use; however, I am proposing that the analysis of its use is a critical piece in demonstrating mathematical proficiency. Further, each content standard should be interrogated for its affordances in particular mathematical scenarios.

It is dangerous to observe Tyler's actions in a *deficient* sense, as a student who simply did not grasp the previous day's lesson. In this light, by failing to employ the Sine Law, Tyler had fallen behind in the game of survival of the fittest. However, through the lens of mathematical fitness, Tyler acted in a *sufficient* manner in order to address the task that was given to him to resolve. His analysis of the possibilities available to him was not based on their place on a continuum of sophistication, but rather on their viability—on their ability to sustain his sense-making. In doing so, he brought together the two groups of mathematical standards in an act of mathematical proficiency.

If we are to honor the commission to enfold content standards in process standards, our focus must shift away from looking for incremental mastery. This involves rejecting the idea of an "absolute fittest," where success is reserved for those who can apply the most sophisticated methods, and embracing a new lens of fitness where "the unit of survival is a flexible organism-in-its-environment" (Bateson, 1972, p. 451). In this regard, this evolutionary language provides an image of synergy between two sets of standards working together to trigger, foster, and extend the fitness of mathematical knowing.

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