Primes, Primitives, and Pythagoras

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Abstract: *The authors explore the connections between prime factorizations and primitive Pythagorean triples, investigating special cases of primitive triangles in order to predict when hypotenuse lengths produce more than one distinct triangle. The authors discuss the usefulness of the method in secondary school classrooms.*

Keywords: *Pythagorean theorem, number theory, Pythagorean triples*

1 Introduction

The Pythagorean theorem is a staple of U.S. high school mathematics curricula. In fact, *The Curriculum and Evaluation Standards for School Mathematics* refers to the Pythagorean theorem as "[o]ne of the most important properties in geometry" (NCTM, 1989, p. 113). The Common Core standards highlight applications of the theorem beginning in 8th grade, so almost every student can rattle off $a^2 + b^2 = c^2$. Whether they realize the complexity of this statement is a different story. With strategic planning and facilitation, teachers *can* use the theorem to provide students with opportunities to engage in the mysterious beauty of mathematics by making conjectures, finding patterns, and making convincing arguments (Hart, et. al, 2008).

In the following paper, we don't call for teachers to reinvent the wheel, but rather we emplore them to turn the wheel on its head and examine it from a different angle. In schools, students often interact with the Pythagorean theorem on homework assignments that consist of right triangles with one side length missing. Such assignments teach students to correctly substitute values into formulas and use a calculator to produce answers. We suggest that these same skills can be mastered through rich activities that promote curiosity, excitement, and discovery. Primitive Pythagorean triangles, or PPTs, are just one example of the many applications to this familiar topic that offers students the opportunity to discover mathematics' mysterious beauty.

2 Primitive Pythagorean Triangles Introduced

Stevenson's *Exploratory Problems in Mathematics* (1997) first introduced us to PPTs: that is, right triangles in which all side lengths are relatively prime positive integers. While not commonly encountered in secondary-level curricula, there are a myriad of ways to utilize this special type of triangle in the classroom. For example, middle school students may not be aware of the term "relatively prime," but they can certainly understand the concept as it simply means each side length shares no common factors with the others. Students could examine a list of Pythagorean triples and be tasked with identifying the triangles that fall into the primitive category, thereby displaying comprehension of prime factorizations and what it means for numbers to be relatively prime. After identifying Primitive Pythagorean triangles, students can be encouraged to analyze the resulting list for patterns—a task that aligns with Common Core Practice Standard MP7, *Look for and make use of structure*. Furthermore, formulating and describing such a pattern in a logical and useful manner addresses Common Core Content Standard HSA.SSE.B.3, *Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression*. Such activities open the door to make use of Pythagorean triangles in a middle school algebra course.

3 Posing the Question

In his book, Stevenson (1997) posed the following task: Given one side length, determine the number of Pythagorean triples and primitive Pythagorean triples that exist. We will explore this question another way.

The Two PPT Problem

What hypotenuse lengths will result in two Primitive Pythagorean Triangles (PPTs)?

The first value for which this occurs is the hypotenuse length of 65. This hypotenuse length creates two separate PPTs as shown in Figure 1.

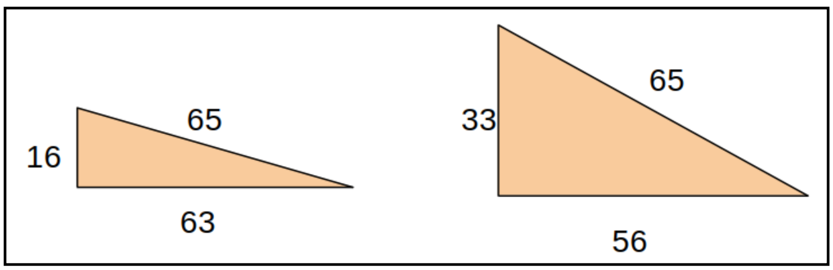

Fig. 1: *The two PPTs with hypotenuse of length 65.*

In an attempt to address our question, we first needed to quickly find all primitive Pythagorean triples, which may be accomplished through Euclid's formula (Moshan, 1959). Moshan showed these triples are generated using two integers of opposite parity, referred to as m and n, which are relatively prime. The side lengths of the triangle are then $m^2 - n^2$, $2mn$, and $m^2 + n^2$. To check that this is true, we take the sum of the squares of the shorter two sides and obtain,

$$\begin{aligned}(m^2 - n^2)^2 + (2mn)^2 &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 && (1)\\ &= m^4 + 2m^2n^2 + n^4 && (2)\\ &= (m^2 + n^2)^2 && (3)\end{aligned}$$

Therefore, the proposed side lengths indeed form a right triangle by the Pythagorean Theorem. Simplifying this expression relied on exponent rules and combining like terms, meaning it is at the level of proficient middle school students. We leave the proof of why the condition of m and n being relatively prime ensures that the triangle is in fact primitive to the reader.

Using this idea, students can use the formula function of Google Sheets to quickly generate side lengths of these triangles (see Figure 2). The blacked out rows correspond to the m and n values that are of opposite parity, but are not relatively prime. In order to organize the information, we decided to sort the spreadsheet by hypotenuse value, C. Figure 3 shows the resulting values.

A	B	C	D	E	F	G	H	I	J
n	m	n^2	m^2	$m^2 - n^2$	2mn	$m^2 + n^2$	A	B	C
1	2	1	4	3	4	5	3	4	5
1	4	1	16	15	8	17	15	8	17
1	6	1	36	35	12	37	35	12	37
1	8	1	64	63	16	65	63	16	65
1	10	1	100	99	20	101	99	20	101
1	12	1	144	143	24	145	143	24	145
2	3	4	9	5	12	13	5	12	13
2	5	4	25	21	20	29	21	20	29
2	7	4	49	45	28	53	45	28	53
2	9	4	81	77	36	85	77	36	85
2	11	4	121	117	44	125	117	44	125
2	13	4	169	165	52	173	165	52	173
3	4	9	16	7	24	25	7	24	25
[redacted]									
3	8	9	64	55	48	73	55	48	73
3	10	9	100	91	60	109	91	60	109
[redacted]									
3	14	9	196	187	84	205	187	84	205

Fig. 2: *The table of PPTs generated and sorted using m, n values.*

	A	B	C	D	E	F	G	H	I	J
1	n	m	n^2	m^2	$m^2 - n^2$	2mn	$m^2 + n^2$	A	B	C
2	1	2	1	4	3	4	5	3	4	5
3	2	3	4	9	5	12	13	5	12	13
4	1	4	1	16	15	8	17	15	8	17
5	3	4	9	16	7	24	25	7	24	25
6	2	5	4	25	21	20	29	21	20	29
7	1	6	1	36	35	12	37	35	12	37
8	4	5	16	25	9	40	41	9	40	41
9	[redacted]									
10	2	7	4	49	45	28	53	45	28	53
11	5	6	25	36	11	60	61	11	60	61
12	1	8	1	64	63	16	65	63	16	65
13	4	7	16	49	33	56	65	33	56	65
14	3	8	9	64	55	48	73	55	48	73
15	2	9	4	81	77	36	85	77	36	85
16	6	7	36	49	13	84	85	13	84	85
17	5	8	25	64	39	80	89	39	80	89
18	4	9	16	81	65	72	97	65	72	97
19	1	10	1	100	99	20	101	99	20	101
20	3	10	9	100	91	60	109	91	60	109

Fig. 3: *The table of primitive Pythagorean triples ordered by hypotenuse length.*

Analyzing the table in this format shows that certain numbers appear twice in the C column, meaning that they are the hypotenuse of two distinct PPTs (e.g., 65, 85). While this seemed intriguing, we felt that only examining PPTs did not tell the full story. As such, we next found a table that listed all Pythagorean triples with hypotenuse less than or equal to 2,100, a partial listing of which is found in Table 1.

Table 1: A partial table of Pythagorean triples ordered by hypotenuse.

Leg 1	**Leg 2**	**Hypotenuse**	**Primitive or multiple of a primitive?**
36	48	60	$(3, 4, 5)$
11	60	61	*primitive*
39	52	65	$(3, 4, 5)$
25	60	65	$(5, 12, 13)$
16	63	65	*primitive*
33	56	65	*primitive*
32	60	68	$(8, 15, 17)$
42	56	70	$(3, 4, 5)$
48	55	73	*primitive*
24	70	74	$(12, 35, 37)$
45	60	75	$(3, 4, 5)$
21	72	75	$(7, 24, 25)$
30	72	78	$(5, 12, 13)$
48	64	80	$(3, 4, 5)$
18	80	82	$(9, 40, 41)$
51	68	85	$(3, 4, 5)$
40	75	85	$(8, 15, 17)$
13	84	85	*primitive*
36	77	85	*primitive*

After examining Table 1, we noticed that if two PPTs appear next to each other, (i.e., there is one hypotenuse length with two possible PPTs), then there also exist two non-primitive Pythagorean triangles with that same hypotenuse. For example, scanning column 4 of Table 1, the only times two PPTs appear in succession is with hypotenuse lengths 65 and 85. Thus, these are the only two cases in this table of numbers that are the hypotenuses of two PPTs and two non-PPTs.

4 Primitive and Primes?

To explore whether there was a pattern among these hypotenuse values, we decided to evaluate the prime factorization of each. This seemed appropriate because prime factorizations are the basis of primitive triangles. We located the first fifteen instances of this phenomenon and compiled the hypotenuse lengths and their prime factorizations in Table 2.

Table 2: The prime factorization for each hypotenuse.

Hypotenuse Length	**Prime Factorization**
65	$5 \cdot 13$
85	$5 \cdot 17$
145	$5 \cdot 29$
185	$5 \cdot 37$
205	$5 \cdot 41$
221	$13 \cdot 17$
265	$5 \cdot 53$
305	$5 \cdot 61$
325	$5^2 \cdot 13$
365	$5 \cdot 73$
377	$13 \cdot 29$
425	$5^2 \cdot 17$
445	$5 \cdot 89$
481	$13 \cdot 37$
485	$5 \cdot 97$

From Table 2, we determined that each prime factorization included only two distinct primes. Furthermore, they are composed of only certain primes. For example, prime numbers 5, 13, 17, and 29 appear several times, while 2, 3, 7, and 11 never appear. We conjectured that once a prime is included in the factorization of one primitive hypotenuse, multiplying it by any other prime from our list gives another example of this phenomenon. Checking with our larger spreadsheet, it appears that this pattern continues to hold.

From this, we sought to find a pattern that could predict what primes are included on this list. Using the Division Algorithm, we noted that all prime numbers greater than five must be of the form $6n+1$ or $6n-1$. We leave the reader and their students to complete the specifics of this proof. Table 3, displays the primes we know at this time in this form.

Table 3: Our list of primes rewritten in $6n \pm 1$ form

Prime Number	**$6n \pm 1$ form**
5	$6(1) - 1$
13	$6(2) + 1$
17	$6(3) - 1$
29	$6(5) - 1$
37	$6(6) + 1$
41	$6(7) - 1$
53	$6(9) - 1$
61	$6(10) + 1$
73	$6(12) + 1$
89	$6(15) - 1$
97	$6(16) + 1$

Here, a pattern began to emerge. Middle and high school students alike could perform this same task of examining the table and looking for patterns. Teachers could use this as a warm-up to spark students' interests as they simply are asked to write down things that they notice and wonder about the tables. In doing this, teachers help students develop a mathematical mindset. We noticed that every time the n value is even, we add one. Every time the n value is odd, we subtract one. To check this property, we rewrote all prime numbers less than 100 that are not included on our list in this form. Here, we see the opposite pattern. If n is even, we subtract one (e.g., $71 = 6(12) - 1$). If n is odd, we add one (e.g., $79 = 6(13) + 1$). Thus, it seems that a prime can be in the factorization of a hypotenuse that generates two PPTs if and only if it can be written in the form six times an even number plus one or six times an odd number minus one. Therefore, to determine whether a number could be the hypotenuse for two PPTs, we must first find that number's prime factorization. Then, we check whether each prime in that factorization is of the aforementioned form. At this point, the pattern seems to be clear, but why does this work?

5 Fermat's Theorem on the Sum of Two Squares

After many attempts to prove this pattern, we realized we could rely on a well-known theorem: Fermat's theorem on the sum of two squares. The theorem states that an odd prime is the unique sum of two squares if and only if it is equivalent to one mod four, meaning that there is a remainder of 1 when the integer is divided by 4.

Fermat's theorem on the sum of two squares

$p = x^2 + y^2$ iff $p = 1(\text{mod } 4)$, where p is prime and x, y are integers.

While the proof of this theorem is beyond the abilities of most middle and high school students, they can use the theorem to prove a number of corollaries. Applying the theorem to our work, each prime p may be considered the hypotenuse of a Pythagorean triple. Analyzing a list of Pythagorean primes (e.g., 5, 13, 17, 29, 37, 41, $\ldots$), we see it is the same list of primes from Table 3. Therefore, we must prove that our version is equivalent.

The Division Algorithm states that any integer a can be written as a multiple of any other integer b plus a remainder r, where the remainder is greater than or equal to zero and less than b. Thus, any integer can be written as a multiple of 3 (denoted $3n$) and a remainder. By the division algorithm, the remainder can take the values of 0, 1, or 2. Thus, all integers are of the form $3n$, $3n + 1$, or $3n + 2$. Consider each case. First, $3n$. Substituting into $4n + 1$ (i.e., 1 mod 4), we obtain

$$4(3n) + 1 = 12n + 1 = 6(2n) + 1, \tag{4}$$

i.e., six times an even number plus one. Next, for $3n + 1$, we have

$$4(3n + 1) + 1 = 12n + 5 = 6(2n + 1) - 1, \tag{5}$$

i.e., six times an odd number minus one. However, we have one last case. Starting in the same way as the previous cases, we obtain

$$4(3n + 2) + 1 = 12n + 9 = 3(4n + 3), \tag{6}$$

indicating this case produces only composite numbers and can therefore be eliminated. Thus, by re-expressing our previous statements, we see that six times an even number plus one and six times an odd number minus one is equivalent, but more concisely written as $4n + 1$. This exercise could be well used in an algebra course as a way of practicing the use of the distributive property and combining like terms. Each of these are imperative to other topics of algebra such as solving linear equations. Using an example such as this provides application to the problem and encourages students to work on mathematical precision in their simplifying.

Thus, any prime of the form $4n + 1$ produces exactly one PPT because of the uniqueness part of Fermat's theorem on the sum of two squares. However, if we compose two of these primes, say $4n + 1$ and $4m + 1$, we obtain

$$(4n + 1)(4m + 1) = 16mn + 4n + 4m + 1 = 4(4mn + n + m) + 1, \tag{7}$$

another number of this form, since $4mn + n + m$ is an integer. Thus, if a composite number is of the form $4n + 1$, and all primes in its factorization are of the form $4n + 1$, then there exist more than one PPT with that number as the hypotenuse. Specifically, if there are exactly two distinct primes in its factorization, then there are exactly two PPTs and two non-primitive Pythagorean triangles with that hypotenuse.

6 Extending the Task

To further explore Fermat's theorem on the sum of two squares, students could prove several lemmas that help build up to Fermats theorem and the corollaries of it.

- If a prime p is of the form $4n + 1$, then p is the hypotenuse of a PPT.
- If h and k are relatively prime hypotenuses of two PPTs, then the product hk is the hypotenuse of another PPT.
- If p is a prime of the form $4n + 1$, then pn is the hypotenuse of a PPT for all integers n greater than or equal to one.
- A number n is the hypotenuse of a non-primitive Pythagorean triangle if and only if it has a prime factor of the form $4n + 1$.

Students could also explore various properties of PPTs. In each example, it must be that the hypotenuse and exactly one leg are odd. Why is this? In addition, the area must be divisible by six. Do any additional properties on the area appear when analyzing just PPTs that share a hypotenuse with another? What patterns do their prime factorizations reveal?

7 Relevance to the Current Affairs of The Classroom

As stated previously, PPTs are not a standard part of the high school curriculum, but they can serve an important purpose. In the article "Bringing Pythagoras to Life," the authors state that, "Students know the theorem by name and can recite $a^2 + b^2 = c^2$ but that they often cannot handle even simple computations using the formula" (Ericksen et. al., 1995, p. 744). This observation suggests that broadening students' experience with the theorem is essential. Introducing PPTs is one way to do this that sparks the curiosity of students leading to increased motivation. In turn, students will be more likely to conceptualize the math causing a more comprehensive understanding.

According to Hlavaty (1959), in high school classes, we "have succeeded in doing three things: first, in driving away from geometry droves of pupils who couldn't take it; second, in boring great numbers in the middle-ability groups and giving them a distaste for mathematics; third, in wasting the time of the capable students" (p. 116). Though written over 50 years ago, these words still ring true today. Instead of simplifying the geometry curriculum, we must engage students in math that allows their creativity and curiosity to flourish. As a solution to this problem, Hlavaty expects teachers to "find the time and the energy to refurbish [their] own mathematical preparation" (p. 118). We claim that this process does not have to be grueling, but can be achieved through a more natural approach to mathematics.

8 Final Thoughts

Overall, students and teachers alike would benefit from exploring this different approach to a relatively common subject. Our exploration was fueled by shifting our focus from generating Pythagorean triples to analyzing patterns within them. Instruction that follows a similar shift in focus has the potential to support teachers in promoting curiosity-creating environments for their students. By allowing students to create their own observations and conjectures, teachers are able to reinforce students naturally occurring wonderings, which may result in an increased student motivation. Furthermore, allowing students to engage in pattern-finding activities with triangles provides students with the opportunity to develop their number sense, which has benefits spanning all areas of mathematics.

We urge readers to consider having their students engage in a rich tasks such as the one described in this report. We believe that these types of tasks will help students of all ability types. By allowing students to think independently and authentically, teachers open the door for productive and exciting math classes.

References

Ericksen, D., Stasiuk, J., & Frank, M. (1995). Bringing Pythagoras to Life. *Mathematics Teacher 88*(**9**), 744–747.

Hart, E. W., Martin, W. G., & Hirsch, C. (2008). Standards for High School Mathematics: Why, What, How? *Mathematics Teacher 102*(**5**), 377–82.

Hlavaty, J. (1959). The Nature and Content of Geometry in the High Schools. *Mathematics Teacher 52*(**2**), 115–118.

Moshan, B. (1959). Primitive Pythagorean Triples. *Mathematics Teacher 52*(**7**), 541–45.

National Council of Teachers of Mathematics (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers (2010). *Common Core State Standards Mathematics*. Washington D.C.: Author.

Pythagorean triples. Accessed 19 April 2018 at http://www.tsm-resources.com/alists/trip.html

Stevenson, F. W. (1997). *Exploratory Problems in Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

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