Making Students' Mathematical Arguments Explicit

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Abstract: *Elementary students are expected to construct viable arguments with increasing sophistication as they progress through schooling. In this article, the authors describe students' progression as different shifts from tacit to explicit mathematical talk and writing. In addition, the authors include a resource for teachers to aid in evaluating and understanding childrens argumentation in grades K–5.*

Keywords: *argumentation, proof, sense making, discourse*

1 Introduction

The widespread adoption of the *Common Core State Standards for Mathematics* (CCSS-M: CCSSI, 2010b) has contributed to increased calls for students to engage in argumentation, justification, and proof. This is evident in CCSS-M's Standard for Mathematical Practice #3 which calls for students to construct viable arguments and critique the reasoning of others (CCSSI, 2010b). The growing consensus in various state standards and among mathematics educators is that argumentation is an important form of mathematical discourse, whether it occurs in written or spoken form (Cirillo et al., 2016). In this article, we provide a practical resource stemming from research with children across grades K–5, namely, a framework for assessing the precision within a child's mathematical talk or writing. We highlight the use of the framework through the analysis of illustrative examples from K–5 students' mathematical argumentative writing.

2 Mathematical Argumentation

2.1 What is Mathematical Argumentation?

We define mathematical argumentation as a process of validating a claim. When students engage in argumentation, they are conveying their mathematical experiences. How students convey these experiences matters significantly. The CCSS-M identifies argumentation as a key standard for mathematical practice, specifying that students should "justify their conclusions, communicate to others, and respond to the arguments of others" (CCSSI, 2010b, p. 6). Similarly, the English Language Arts standards across K-5 specify that students should make claims, establish these claims with facts and details, and provide reasons for these various aspects of their writing (CCSSI, 2010a). Indeed, argumentation is a genre that crosses multiple content areas (Toulmin, 2003). However, the distinction between mathematical argumentation and other forms of argumentation is not always clear, particularly in the elementary grades.

2.2 Features of Argumentation

Since argumentation is a process, it can be difficult to identify where in this process elementary aged children are engaging. Essentially, a mathematical argument involves building off of given information in a task to propose statements that support a claim. These statements that connect the given information with the claim go by many names (warrants, propositions, etc.). Some of these statements serve simpler roles of moving from one point in an argument to another, while others do a bit more (i.e., justifying why some statements are appropriate). Complicating things further is that some children's statements are less precise than others' and include some, but not all, parts of a complete explanation.

2.2.1 A classification scheme

To aid in teasing apart these various features, we use a classification scheme for mathematical argumentation (Kosko & Zimmerman, in press) which includes mathematical recounts, procedures, descriptions, and explanations. Children who engage in mathematical recounts will tell you what they did when engaging in a math task, but they are not consistently explicit about the mathematics. For example, in an addition task, such children may say that they put numbers together, or that they combined objects, but may not say the actual numbers that were added, or will not specify addition. The mathematical information is tacit because most knowledgeable teachers will be able to derive what the child did by what the child says, writes, or draws. By contrast, children who provide a mathematical procedure are explicit about the numbers they use and the actions they take (e.g., addition, subtraction). However, the justification for these actions is tacit, in that the justification can be inferred from what the child says or does but is not explicitly stated. By contrast, a mathematical description provides a clear, explicit justification. Some children go further and provide a mathematical explanation, which includes a justification that explicitly identifies a mathematical rule, definition, or property (e.g., definition of odd numbers, commutative property, etc.). Children who provide mathematical descriptions instead of explanations may need scaffolding to help them be more explicit. However, it is important to remember that not every task prompts students for such explicit information.

2.2.2 Student work samples

Figure 1 includes example responses of each classification for two different tasks. For recounts, each example conveys that the child did something mathematically; however, this mathematics is not explicit. For instance, student work in Figure 1 indicates that a proposed strategy is correct "because he's stealing." A teacher may recognize that the child is referencing adding 2 to $19 + 19$ and then subtracting the 2 again with $20 + 20 = 40$. However, none of the mathematical actions are explicitly conveyed by the work sample either in writing or pictorially.

By contrast, mathematical procedural writing conveys explicit mathematical actions, but justifications are either absent or not explicit. For instance, in the first procedure example in Figure 1, the student explicitly writes what was added and how 38 was found, but the student does not relate 38 to adding or subtracting the 2 in Omar's strategy. By contrast, the student's description in the next example explicitly notes Omar's adding/subtracting of 2 as a justification for why the strategy works. Although no example of a mathematical explanation was observed in our work with this group of students, it is possible that a child could justify Omar's strategy explicitly by citing the Associative Property by name, or by an alternative name often used by textbooks (e.g., "Break Apart Strategy"). Thus, one can see how providing more specific information both in regards to procedures and their justifications can be observed in students' mathematical writing.

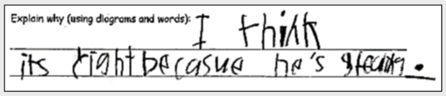
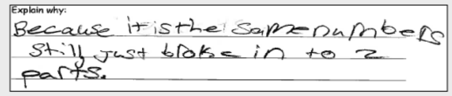
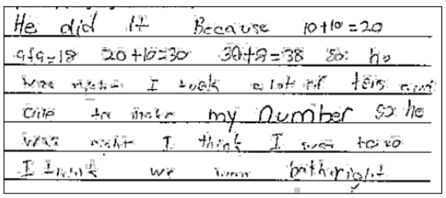
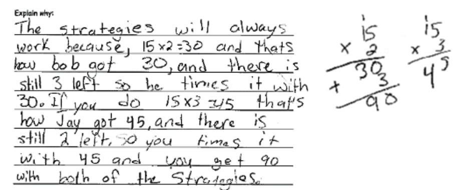
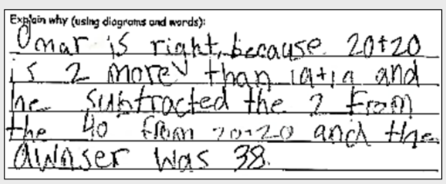
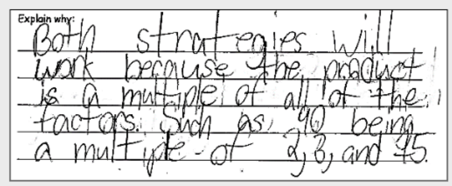
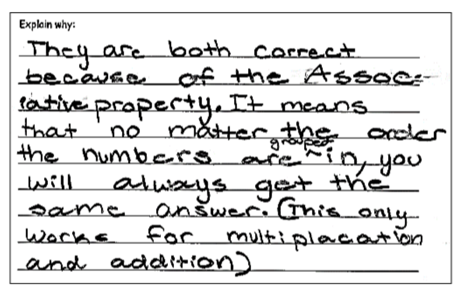
	Task 1 Examples:	Task 2 Examples:
	Omar was asked to find the answer for 19 + 19. This is what he wrote on his paper: 19 + 19 20 + 20 = 40 40 − 2 = 38 Explain how Omar solved the problem and why it does (or doesn't) work.	Jay and Bob were asked to solve 3 x 15 x 2. Both showed their work below: Jay wrote: 45 x 2 = 90 Bob wrote: 3 x 30 = 90 Both Jay and Bob solve the problem correctly. Explain why both strategies will always work for problems like 3 x 15 x 2.
Recount	 "I think it's right because he's stealing."	 "Because it is the same numbers still, just broke into 2 parts."
Procedure	 "He did it because 10 + 10 = 20, 9 + 9 = 18, 20 + 10 = 30, 30 + 8 = 38. So he was right. I took a lot of tens and ones to make my number. So he was right, I think I was too. So I think we was both right."	 "The strategies will always work because, 15x2 = 30 and that's how Bob got 30, and there is still 3 left so he times it with 30. If you do 15x3 = 45 that's how Jay got 45, and there is still 2 left, so you times it with 45 and you get 90 with both of the strategies."
Description	 "Omar is right because 20 + 20 is 2 more than 19 + 19 and he subtracted the 2 from the 40 from 20 + 20 and the answer was 38."	 "Both strategies will work because the product is a multiple of all of the factors. Such as, 90 being a multiple of 2, 3, and 15."
Explanation		 "They are both correct because of the associative property. It means that no matter the order the numbers are grouped in, you will always get the same answer. (This only works for multiplication and addition)."

Fig. 1: *Student Examples from Two Mathematical Writing Tasks.*

The two tasks in Figure 1 prompt students for different kinds of information. For Task 1, second and third grade students were asked to explain why Omar's strategy did or did not work. For Task 2, fourth and fifth grade students were asked to explain why both Jay and Bob's strategies will always work. Each task asks for a justification, and both omit key steps in the displayed student work in order to press responding children to specify what may be missing (a strategy we have found to be fairly effective when constructing tasks). However, Task 2 is structured in a way that is more likely to yield a justification that cites a mathematical property (e.g., the associative property of addition). This is neither good nor bad, as it depends on the teacher's goals for a particular lesson or assignment. We draw attention to it here because the way we structure our tasks influences how children engage in mathematical argumentation.

Although differences between some types of argumentation may seem subtle, these differences are important for differentiating between children's mathematical arguments. With practice, most teachers can use the classifications framework to quickly and informally identify how explicit a child's mathematical argument is. Then, they can use follow-up questions and other scaffolding techniques to encourage the child to convey more mathematical meaning, specifically focusing on the structure of the mathematics involved in a task. For example, in the child's recount in Figure 1 that identified Omar was "stealing," a follow-up prompt could ask the child to explain what this stealing looks like, and to encourage the child to use the numbers involved, draw a diagram, etc.

2.3 Strategies for Eliciting Student Writing

In our work engaging students with mathematical writing, we have developed three rules of thumb for creating tasks that elicit more detailed writing. First, if a task is too simplistic, students' responses will also be simplistic. Second, when a task focuses on the structure of number, students tend to write in more detail. One example of this involves some application, formally or informally, of the Associative Property. For example, Jay and Bob's strategies in Figure 1 are each informal applications of associativity. The relationship between Jay and Bob's strategies can be rewritten more formally as $3 \times (15 \times 2) = (3 \times 15) \times 2$. We have also observed using open number sentences with two equivalent expressions to be useful prompts. For example, asking students to solve $22 + 19 = 14 + \square$ without adding the total on either side encourages composing and decomposing number to relate the structure of numbers on each side of the equation to find a solution. Third, we provide examples of student solution strategies that we ask children to explain or critique (such as those provided in Figure 1). Although the strategies provided in Tasks 1 and 2 in Figure 1 are correct, we have also found incorrect strategies to be useful for eliciting critique and discussion from students.

3 Tacit versus Explicit Justifications

We have found the classification scheme illustrated in Figure 1 to be helpful in scaffolding students' mathematical argumentation to be more explicit and precise. However, students don't always talk or write in ways that are as clear as the examples in the figure. This is especially true when students attempt to justify their mathematics. Justifications are, at their core, an accepted reason for doing something. According to Ellis (2007), accepted mathematical reasons relate to some facet of generalizing activity, such as examining relationships, deriving a mathematical rule or definition, or applying the mathematics to a new setting. So, justifications that examine relationships may be conveyed tacitly through relating mathematical objects or explicitly through identifying the relationship. Similarly, justifications that focus on a mathematical rule or definition can be conveyed tacitly through a description of searching via multiple examples or through stating a mathematical definition, property, or rule (Ellis, 2007). Given the different kinds of justification and the different ways each can be conveyed (tacitly or explicitly), it is understandable that the differences between tacit and explicit justifications often go unnoticed. Further, as educators we see mathematical meaning in what children do and how they do it—even when our students fail to do so.

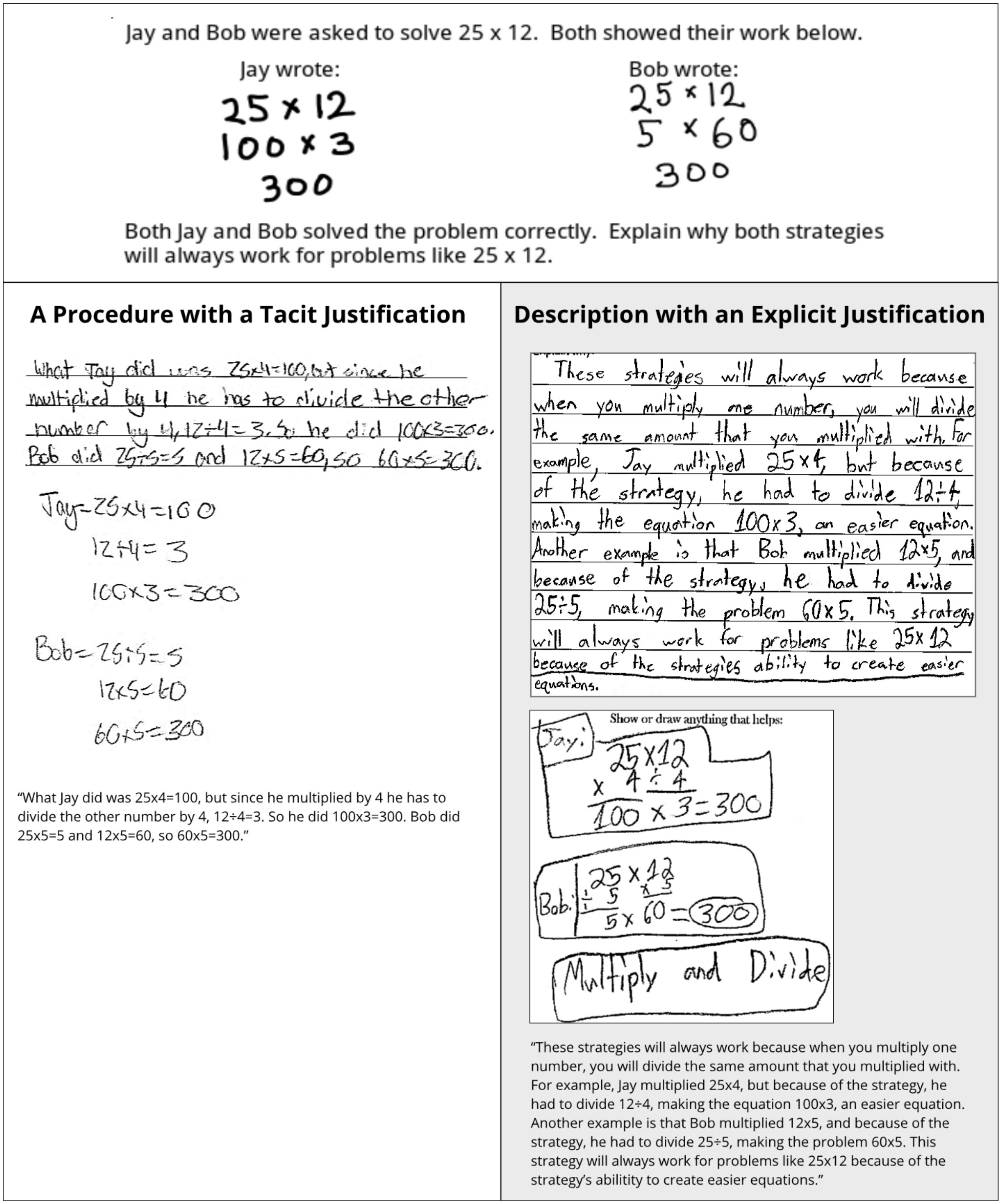


Fig. 2: *Examples of Tacit and Explicit Justifications.*

Learning to engage in mathematical justification is a difficult process for students, but it is also an essential aspect of learning (Balacheff, 1988; Reid, 2002; Store, 2015). Because it can be a difficult process for students, it can also be difficult for teachers to interpret whether a student is providing a justification that is mathematically sufficient. Consider the two fourth grade students' responses in Figure 2. In each child's work displayed with their writing, they include the same representation that shows an exchange of multiplying and dividing by the same number (for both Jay and Bob). However, each child's writing conveys this work differently.

The child who included a tacit justification wrote down the steps that Jay and Bob took to find the same answer through different strategies. The child even justifies Jay's strategy by stating that "since he multiplied by 4 he has to divide the other number by 4" but did not do so for Bob. Thus, the child effectively is using this justification as a way to connect the given information to her claim (Kosko & Zimmerman, in press). By contrast, the child who included an explicit justification also wrote down the steps that Jay and Bob took to find the same answer through their strategies. However, this child goes further with justifying by stating that "when you multiply one number, you will divide the same amount that you multiplied with." Although this statement can be revised to be more precise, it effectively states a mathematical rule (Ellis, 2007), even if not by a proper name. Additionally, the procedures this child describes are included as examples of the justification, not as a means of conveying the procedure.

4 Discussion

The classifications discussed in this article help to illustrate one way that teachers can evaluate students' mathematical argumentation. It is important to note that these classifications are useful only when a teacher is seeking to engage students in some form of mathematical argumentation. For example, although justifying one's mathematics is important, not all math activities need to engage students in justification. Similarly, it is sometimes perfectly acceptable for a student to provide a mathematical recount, such as during a brief whole class review of a topic. Furthermore, there is preliminary evidence that, depending on the math topic, not all classifications illustrated in Figure 1 should be expected of students at particular ages or developmental stages (Kosko & Zimmerman, in press). Thus, the classification framework is a useful tool, but its use is contingent upon context (both in regards to content and grade level).

The "show and tell" form of mathematical discourse typically involves either students recounting their mathematics or providing a procedure. It is when students are explicit about their justifications that they move beyond show and tell. One means of developing children's justification skills is to encourage using various kinds of examples and scaffold students' generalization of these examples (Reid, 2002; Store, 2015). This can include example strategies of other students (real or fictional), but may also involve tasks that require students to create multiple examples. Another means of developing justification is to provide students with tasks that have a degree of complexity (Kosko, 2016). This involves making sure that the tasks are not too simplistic for students (Kosko, 2016), since students are less likely to see the need to justify something that is easily understood by them or others. If we want students to engage in justification, we need to give them a task that is complex enough to need justifying. This leads to a third recommendation of focusing on the mathematical structure of tasks we provide students. One way we sought to do this was to focus on tasks where mathematical properties are pragmatically useful (e.g., Jay and Bob's informal use of Associative Property in Figure 1). Another way to ensure a task attends to mathematical structure is to focus on concepts we know are conceptually difficult for students, such as understanding what the equals sign means, regrouping in addition/subtraction, fraction multiplication or division, and so forth. By focusing on the mathematical structure of more conceptually challenging topics, students engage in mathematical argumentation in a meaningful manner.

In order to support students' meaningful engagement in mathematical argumentation, we need to meet students where they are in regards to their demonstrated mathematics. This is true for various content, but particularly so for mathematical discourse. The classifications framework presented in this paper provides a pragmatic tool to distinguish between different kinds of student discourse for both written and spoken mathematics.

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