Rich Task Construction: Making "Good" Problems Better

Michael Waters *Northern Kentucky University*

Abstract: *The author describes characteristics of rich problems and provides mathematics teachers with strategies for transforming routine exercises into problems that promote problem-solving practices with a focus on problems for which a solution is not immediately apparent. Several problems and solutions are presented with a mindful approach to assigning problems.*

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1 Introduction

Textbook tasks abound in the mathematics classroom. Often, mathematics teachers are faced with the difficult choice of which tasks to assign to students. Exercises focus on important skills that students must master, while problems are designed to be more conceptual in nature or to hone necessary reasoning and problem-solving practices. Moreover, problem-solving in mathematics involves students "engaging in a task for which the solution method is not known in advance" (NCTM 2000, p. 52). Mindful teachers struggle to find the appropriate balance of exercises and problems for their students. A rich mathematics problem is one that has "a range of characteristics that together offer different opportunities to meet the different needs of learners at different times" (Piggot 2011, par. 2). The purpose of this manuscript is to suggest characteristics of rich mathematical problems and to assist teachers in making mathematics exercises richer for their students.

2 Characteristics of Rich Problems

Rich problems have been characterized in several sources, such as in the Secondary National Strategy for School Improvement (National Department for Education 2014). While not an exhaustive list, the following are indicators of rich problems; that is tasks that can meet the needs of different learners at different times.

1. Mathematically Engaging
2. Extendable
3. Multiple Entry Points
4. Make Connections
5. Appropriate Use of Tools
6. Multiple Ways of Thinking
7. Problem Solving Heuristics
8. Justification
9. Non-Intuitive and Surprising
10. Encourage Creativity and Originality
11. Communication
12. Develop Confidence and Increase Self-Efficacy

We describe these indicators in more detail in the following paragraphs.

1. Rich problems are *mathematically engaging* That is, they spark student interest in mathematics. Often this refers to problems that relate to students' interests or intrigue students in some way.

2. Rich problems are *extendable*. This means that either the problem itself or its method of solution can be generalized to other situations. Problems like this are instructive and allow students to develop a catalogue of tools through which they can solve future problems.

3. Rich problems lend themselves to *multiple entry points*. That is, there are several ways in which a student can attempt a solution. Often, trying an approach or strategy will lead to a spark of ingenuity—an "aha moment"—which can allow students to proceed.

4. Rich problems allow students to *make connections*. That is, they allow students to see connections within mathematical topics, across the mathematics discipline, and to the real world.

5. Rich problems allow students to *make appropriate use of tools*, including technology. Technology can be useful for not only understanding problems, but to make problems more accessible to younger learners.

6. Rich problems engage students in *multiple ways of thinking*. Sometimes, problems can be solved using strategies that are promoted in the primary grades, but often discouraged thereafter. For example, the "guess and test" strategy can often lead students into thinking about a problem in a different way.

7. Rich problems require the use of *problem-solving heuristics*. This means that, although a solution method is not immediately apparent, it is within the realm of the students experiences to develop one or use a previously learned strategy.

8. Rich problems allow for *justification*. Part of the joy of doing mathematics is to be able to answer the question "why?" Asking for justification can solidify and confirm students' solutions.

9. Rich problems are often *non-intuitive* and *surprising*. These type of problems can generate student engagement in mathematics and can serve to foster student investment in problem-solving.

10. Rich problems *encourage creativity and originality*. After all, the method of solution to such problems is not immediately apparent. So, the struggle to develop appropriate strategies is an important metacognitive process. Teachers should be cognizant that student learning happens as part of that struggle.

11. Rich problems offer students the opportunity to *communicate* effectively. This will allow students to regard answering mathematics problems as a procedure, rather than an end. A solution itself may do very little to convince someone as to its correctness. The way a solution is presented; therefore, can provide a convincing argument to a solution's validity.

12. Rich problems allow for students to *develop confidence* and *increase self-efficacy*. Being able to justify a solution not only increases a student's understanding of the content, but can increase his or her confidence and self-efficacy about mathematics problem-solving. This affective aspect of problem-solving is often overlooked. When students succeed mathematically, they begin to believe in their own mathematical abilities, making them better problem solvers.

3 Making Mathematics Problems Richer

While authors of some problems might expect specific numerical answers, other authors might desire more open-ended solutions or allow for multiple interpretations. Mathematics teachers should not only pay attention to the content of a problem, but also to its effect on the metacognitive aspects of learners while problem-solving. I've found the following strategies helpful for making the tasks I assign to students mathematically richer. The strategies listed either deal with how problems are written (creation) or how they are enacted (implementation), or both.

1. Take advantage of surprising student solutions (implementation).
2. Remove information from a problem (creation).
3. Change the context of a problem (creation).
4. Break a pattern (creation).
5. Ask for justification (creation/implementation).
6. Allow for the use of multiple tools (creation/implementation).

3.1 The Consecutive Integers task

For instance, consider the following problem:

> **Consecutive integers task**
>
> The sum of six consecutive integers is 447. What are the integers?

A typical student solution by Elena, a high school Junior, is offered in Figure 1.

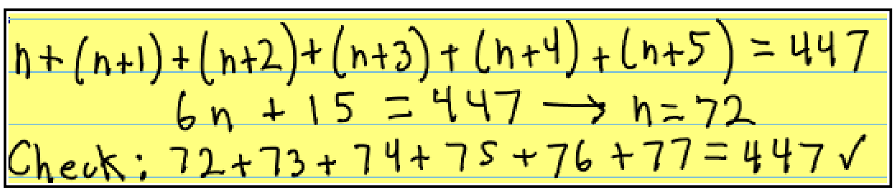


Fig. 1: *Typical student solution to the consecutive integers task.*

Another student J.T. uncovered a "surprise" solution by averaging the six integers (see Figure 2).

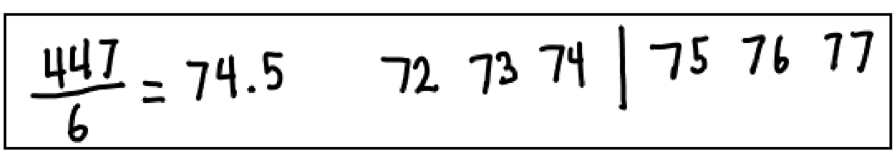


Fig. 2: *Surprise student solution to the consecutive integers task.*

The reader might reflect how often a student presents a surprising, perhaps unintended, solution. Rather than viewing these solutions as anomalies, these are perhaps better seen as opportunities to promote classroom discourse and to uncover student-centered problem-solving strategies. This brings us to the first suggestion for making a mathematics problem richer.

1. Take advantage of surprising student solutions. Rather than viewing surprising solutions as divergent anomalies, consider them as a tool for encouraging student discourse, after students have solved a problem. Student-created strategies are often more intuitive or inventive than teacher-directed strategies and promote metacognitive aspects of problem-solving. Strongly related to a students method of solution is his or her self-efficacy while problem-solving (Waters 2003). Student-created strategies; therefore, can improve confidence and allow students to engage in beneficial discourse while discussing solutions.

A second suggestion for making a mathematics problem richer is:

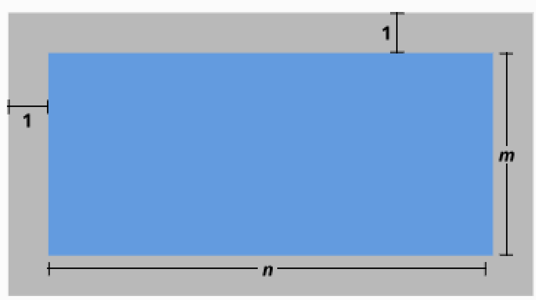
2. Remove information from a problem. Frequently, removing information from a problem makes it more open-ended and subject to different—though equivalent—methods of solution and/or multiple correct answers. Open-ended problems are problems that are "formulated to have multiple correct answers" (Becker and Shimada 1997, p. 1). Student responses to such problems can indicate higher-order thinking and reveal students' abilities to synthesize and apply knowledge to new situations.

3.2 The Swimming Pool task

Consider the following problem:

> **Swimming Pool task**
>
> Suppose you have a rectangular pool of length 25 feet and width 15 feet and you want to build a border around it with 1 foot by 1 foot square tiles. How many tiles will you need?

Removing some information from this problem yields a much richer experience for students:

> **Swimming Pool task**
>
> Suppose you have a rectangular pool of length m feet and width n feet, where m and n are integers, and you want to build a border around it with 1 foot by 1 foot square tiles.
>
> 
>
>
> How many tiles will you need? (NCTM 2000, p. 68)

Note that the number of tiles in the border may be expressed as

$$2m + 2n + 4, 2(m + 2) + 2n, 2m + 2(n + 2), 2(m + n + 2),$$

or other different—though equivalent— answers. This type of problem can lead to classroom discourse about multiple equivalent symbolic representations that describe the same phenomenon and reveal insight into students' ways of thinking about a problem. Another suggestion for making a problem richer is:

3. Change the context of a problem. Changing the context of a problem can not only make it more engaging for students, but can even suggest alternate strategies.

3.3 The Integer Sum task

Consider the following problem:

Integer Sum task

Find the sum of the first 30 positive integers. Find a formula for the sum of the first n positive integers.

The reader might recognize this problem as another classic that appears in many textbooks for future teachers of mathematics. This problem can be given in a different context. For instance, consider changing the problem to the following.

The Handshake Problem

There are 31 people at a party. If each person shakes hands with each other party-goer exactly once, how many handshakes will there be? Generalize your solution.

This change of context might suggest a more active approach to solving the problem. It might suggest strategies such as "act it out" or "solve a simpler problem," often overlooked strategies by novice (and expert!) problem-solvers. Such problems also allow students entry points into a problem that might look, at first, difficult to approach.

One possible solution might involve creating a table representing a simpler problem in order to recognize a pattern and generalize (see Table 1). The pattern might be recognized by noting

Table 1: *Handshake Problem data for various numbers of people.*

People	2	3	4	5	6	7	8	9	10	...	31
Handshakes	1	3	6	10	15	21	28	36	45	...	?

that, if there are n people at a party and another party-goer joins, then n additional handshakes are necessary. So, for 31 people at a party, there are $1 + 2 + 3 + \ldots + 30$ total handshakes. The sum can either be computed directly or by recalling the formula (attributed to Gauss) for the sum of the first n positive integers: $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. In the case of the above problem, $1 + 2 + 3 + \ldots + 30 = \frac{30 \times 31}{2} = 465$. The reader might also notice that the number of handshakes for n people may be represented as diagonals of a regular $n-$gon, with vertices representing distinct people and diagonals representing handshakes between them. Moreover, the number of handshakes in Table 1 forms a sequence of triangular numbers; the general solution to which may be found geometrically.

Another suggestion for making a problem richer is:

4. Break a pattern. Some of the most enjoyable and intriguing problems are those that seemingly break a pattern or whose solutions hold for some, but not all cases.

3.4 The Prime Number Function task

Consider the following problem:

> **Prime Number Function task**
>
> True or False: The formula $p(n) = n^2 + n + 17$ yields a prime for all whole numbers n?

One high school student offered the following clever solution:

$$p(17) = 17^2 + 17 + 17, \text{which is obviously divisible by 17. So, no.}$$

To make the problem more rich, add the phrase: *If not, find the smallest whole number n for which p(n) is not prime.*

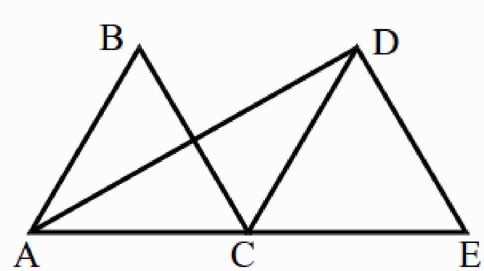
This changes the problem dramatically and induces students to check primality for small whole numbers. In this case, $p(n)$ is prime for $n = 0, 1, 2, \ldots, 15$, but fails when $n = 16$: $p(16) = 16^2 + 16 + 17 = 16(16 + 1) + 17 = 16 \cdot 17 + 17 = 17(16 + 1) = 17^2 = 289$. So, careful choices and wording of such problems can reinforce problem-solving habits of mind by encouraging perseverance and attention to precision while solving problems, key standards for mathematical practice (CCSSI 2010, p. 6). The next suggestion for making a problem better is:

5. Ask for justification. One of the most important questions a teacher can ask his or her students is, "Why?" This question often leads students to use and translate among mathematical representations and to communicate mathematically.

The following geometry problem is typical of many in that students often can get an answer without being able to justify it.

3.5 The Equilateral Triangles task

Consider the following task:

> **The Equilateral Triangles task**
>
> $\triangle ABC$ and $\triangle CDE$ are equilateral with side length 1.
>
> 
>
>
> What is the length of segment $\overline{AD}$.

Consider adding the sentence: *Justify your answer.*

This seemingly simple addition can change the nature of the problem and require expectations of reasoning and proof. One solution offered by Luke is illustrated in Figure 3.

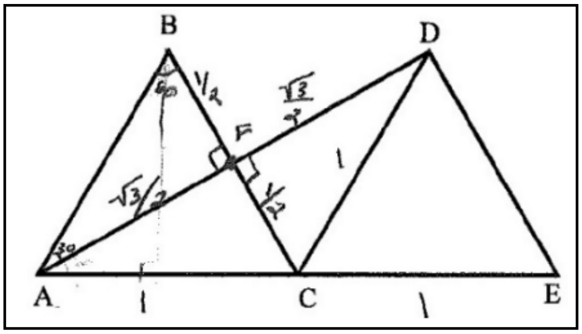


Fig. 3: *A solution to the Equilateral Triangles task.*

Knowing the triangles are equilateral, we can use $\overline{AD}$ as an altitude through $\overline{BC}$ in $\triangle ABC$. Since $\triangle ABF$ is a $30° - 60° - 90°$ right triangle, we know $AF = \frac{\sqrt{3}}{2}$ and $BF = FC = \frac{1}{2}$. Since $\triangle FCD$ is a right triangle and $CD = 1$, the Pythagorean Theorem gives $FD = \frac{\sqrt{3}}{2}$. So, $AD = AF + FD = \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}$. Another suggestion involves the approach to problem-solving with available tools:

6. Allow for the use of multiple tools, especially technology. The author discovered a set of coordinate geometry problems (source unknown) which included the following gem:

> Coordinate Geometry task
>
> The triangle $\triangle ABC$ is isosceles. B is $(-2, -1)$ and C is $(1, -4)$. If the y-coordinate of A is 1, what is the x-coordinate?

A typical (incomplete) student paper-and-pencil solution is provided in Figure 4.

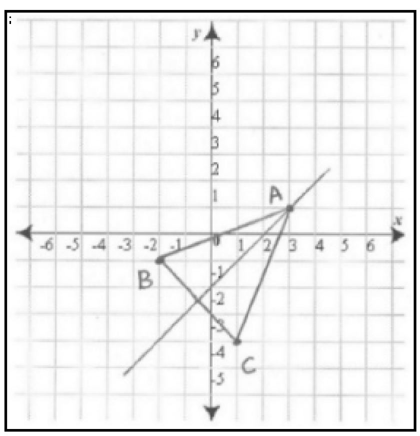
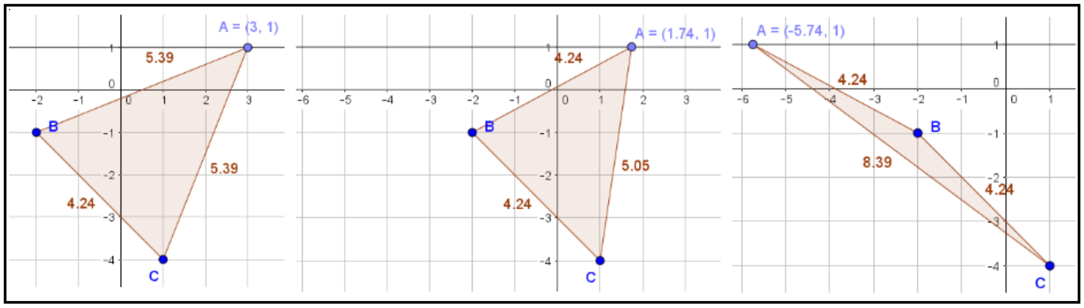


Fig. 4: *A hand-drawn solution of the coordinate geometry problem.*

Assuming A has coordinates $(x, 1)$, the Distance Formula with $AB = AC$ gives $\sqrt{(x-(-2))^2 + (1-(-1))^2} = \sqrt{(x-1)^2 + (1-(-4))^2}$, which yields the quadratic equation $x^2 + 4x + 4 + 4 = x^2 - 2x + 1 + 25$. Collecting like terms and solving gives $6x = 18$ or $x = 3$.

With the availability of dynamic geometry software, Bob, another student in the class, chose to approach the problem with a different entry point using dynamic geometry software and offered the following solution:

Sketch the points B and C, and affix point A to the line $y = 1$, thus fixing the y-coordinate of A to be 1. Sliding point A along the line illustrates several possible solutions, indicating that the triangle could be isosceles in three different ways, as shown in my following sketch.



These solutions may be found by considering three possibilities: $AB = AC$, $AC = BC$, and $AB = BC$ where A has coordinates $(x, 1)$. Note my above sketch indicates one solution if $AB = AC$, no solution if $AC = BC$, and two solutions if $AB = BC$.

If $AB = AC$, the single solution given above yields $x = 3$. If $AC = BC$, the Distance Formula yields the quadratic equation $x^2 - 2x + 8 = 0$. A quick check of the discriminant, $b^2 - 4ac = (-2)^2 - 4(1)(8) = -28$, means the equation has no real solutions. One might also notice that BC is approximately 4.24 units and the distance from C to the line $y = 1$ is 5 units, greater than BC, so no real solution is possible. If $AB = BC$, the Distance Formula yields the quadratic $x^2 + 4x - 10 = 0$. Using the Quadratic Formula, $x = \frac{-4 \pm \sqrt{56}}{2}$, giving approximate values $x = 1.74$ or $x = -5.74$.

Although the wording of the problem implies a single solution, use of dynamic geometry allows students to see multiple correct solutions, leading to a more complete answer.

4 Reflections

Thoughtful teachers pay careful attention to the tasks they assign and are intentional about their problem-solving expectations. Understanding what makes a problem rich assists teachers in wording exercises and examining student solutions to highlight multiple strategies. Rather than searching for—or creating their own—problems, teachers can use these suggestions to transform routine exercises found in many textbooks into rich problems having characteristics that encourage appropriate problem-solving practices and promote classroom discourse about problem-solving.

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Michael Waters, watersm1@nku.edu, is a professor in the Department of Mathematics & Statistics at Northern Kentucky University. His areas of interest include machine learning, and state and national assessments. Mike currently develops virtual interactive learning tools for mathematics teaching.