Problem Solving the Hungarian Way

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Abstract: *Lajos Pósa (pronounced Posha) is one of the most prominent mathematics educators in Hungary. The authors introduce readers to the Pósa method for teaching mathematics, an approach consistent with exploratory methods promoted in the United States. The authors present two scenarios (with extensions) that were posed during the workshop, both with different strategies and goals.*

Keywords: *Problem solving, inquiry, discourse, teaching methods*

1 Introduction

In the summer of 2018, I had the most incredible opportunity to study abroad as a student in the Budapest Semesters in Mathematics Education (BSME) program. I am a secondary mathematics education major at Indiana University of Pennsylvania and was encouraged to apply for this program during my junior year. Because I hope to create future classroom experiences which allow students to learn and discover mathematics in interesting and enjoyable ways, I was excited about this opportunity.

1.1 What is Budapest Semesters in Mathematics Education?

Budapest Semesters in Mathematics Education (BSME) is a study abroad program for undergraduate students and recent graduates who desire to learn more about teaching mathematics (for more information, see page 63). Sixteen applicants were chosen from all over the United States and Canada to attend this 6-week seminar. We spent the first five weeks attending classes in the center of Budapest, which included a crash course in Hungarian, technology in secondary math education, games and manipulatives, and problem-solving using the Pósa method. In the final week of the program, we moved from a dormitory in the city to a home in the mountains of Mátrafüred to join approximately 100 Hungarian children at a summer camp for gifted students from grades 5–9. At the camp, we observed and participated in lessons and activities designed to engage gifted middle and high school mathematicians as we observed teaching methods inspired by Lajos Pósa.

1.2 Who is Lajos Pósa?

Lajos Pósa (pronounced Posha) is one of the most prominent mathematics educators in Hungary. His methods were originally developed for gifted students to discover concepts through individual exploration, but were soon used successfully with many different levels of students. One of the main aspects of his method is to have students experience how professional mathematicians engage with mathematics. Figure 1 shows the 2018 BSME scholars with Lajos Pósa (center). I am the third scholar from the right in the back row.

Fig. 1: *Lajos Pósa with the 2018 BSME scholars.*

The purpose of this paper is to introduce readers to the Pósa method for teaching mathematics. Pósa's vision for how and what mathematics should be taught is valuable for teachers in the United States since it encourages the exploratory methods that support deep student learning consistent with recommendations set forth by the Common Core (2010). In this paper, we present two scenarios posed during the workshop that use a strategy that requires students to follow a sequence of questions to find a solution.

2 Scenario #1: Regions with pie cutting

The first scenario is posed to students as:

> With one straight cut you can slice a pie into two pieces. How many pieces of pie can be created from three cuts? (a cut must be a chord of the circle)

The structure of Pósa's strategy is to engage students in a progression of questions that build upon students' previous knowledge to enable them to develop a general solution.

The most common question from students at the start of this problem, and a logical one after some thought, is: "Do all the pieces have to be the same shape?" If the student does not pose this question, then the teacher can step in to offer it. The goal of the teacher's line of questioning is to create an internal monologue in the student that mimics the teacher. Teachers encourage this gradual trade-off to student questioning by asking, "What do you think I will ask you next?" This anticipation of future questions becomes embedded in the discovery process, with students ultimately developing their own sequence of questions without the aid of the teacher. Many times, students start out with a solution similar to that shown in Figure 2, with all lines intersecting at the center of the circle.

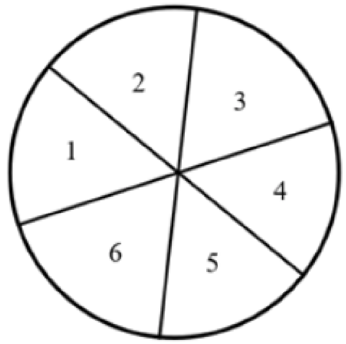

Fig. 2: *A common student solution.*

Although the pie-cutting scenario has one specific answer, it is meant to be the springboard for a sequence of follow-up questions. For example, once students have offered an initial answer, the teacher may ask, “Is this the greatest number of pieces that can be created from three cuts? How do you know?” Students then must develop a logical defense for their claim. The advantage of posing such a problem to a class of students is that there will likely be one or more students who discover that a pie can be cut into seven pieces. If students do not reach this conclusion on their own, offer gentle hints like “Try to see if you can cut it so that there is one more piece.” This same question can be offered even after they have shown a pie can be cut into seven pieces. Students must then prove that no more than seven pieces can be created, which leads into the next question of how many pieces can be shown with four cuts.

Explanation: With three straight cuts through the center of the circle, a pie can be cut into six equal slices. Since the slices do not necessarily need to be equal, the pie can actually be cut into more. One cut divides the pie into two regions. The second cut makes at most two additional regions, splitting the pie into at most four regions. The third cut divides the pie into at most three more regions. Therefore, the greatest number of regions made by three cuts is $2 + 2 + 3 = 7$ regions. A line can divide a plane into at most two half-planes. To create the maximum number of regions with each line in the circle, each line must divide as many regions as possible into half-planes. This is achieved by drawing each new line to intersect all previously drawn lines. The result is a series of cuts similar to those shown in Figure 3.

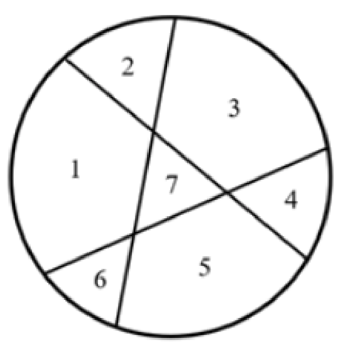

Fig. 3: *One possible diagram of a pie cut into seven pieces.*

Once students understand the first scenario, they are then asked, “How many regions can be created from four cuts, five cuts, six cuts?” These scenarios are significantly more difficult to draw and encourage different problem-solving strategies including the use of non-pictorial representations. If students are not already doing so, the teacher may suggest the use of a table consisting of number of cuts and corresponding numbers of regions. Once the table is complete, it is relatively easy for students to recognize that the pieces of pie are increasing in a pattern as suggested in Figure 4.

Number of cuts (n)	Maximum number of regions created (p_n)	Change in number of regions from last cut
0	1	n/a
1	2	+1
2	4	+2
3	7	+3
4	11	+4
5	16	+5
6	22	+6

Fig. 4: *Maximum number of regions created by n cuts.*

Ask students to continue this table a few more rows and ask them about what patterns they see. Teachers may wish to encourage students to create an equation which describes the maximum number of regions created as a function of the number of possible cuts. If students have not had experience before with recursive sequences, this would be an optimal time to introduce them to terminology and notation. One form of this recursive sequence is:

$p_n = p_{n-1} + n$, with p_n the maximum number of regions made with n cuts (with $p_0 = 1$).

Ask students to relate this equation to the context of the original problem. One possible response is, "To find the number of regions that can be made from n cuts, add together the number of regions from the previous cut (p_{n-1}) and the number of cuts to be made, n." Therefore six cuts can make:

$p_6 = p_{6-1} + 6 = p_5 + 6 = 16 + 5 = 22$ regions.

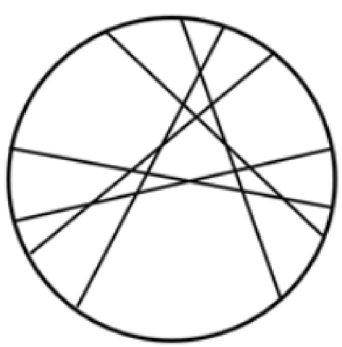

Fig. 5: *One possible diagram of a pie with six cuts (22 regions).*

In order to differentiate for more advanced or more curious students, the teacher can ask "How many regions can be created from thirty cuts?" Students quickly realize that a recursive sequence isn't an efficient tool to answer such a question. The question prompts them to search for an explicit equation for the relationship ($p_n = \left(\frac{1}{2}\right)\left(n^2 + n + 2\right)$ which yields 466 pieces for 30 cuts). The proof of this equation lies in identifying and using series, which are not taught until higher-level calculus courses, however a teacher might want to challenge certain students to try to prove it. The proof can be found at: `http://mathworld.wolfram.com/CircleDivisionbyLines.html`

A key component of the Pósa method is not for students to prove concepts, but for students to have an experience before formality; time to develop personal connections with the mathematics before defining them in a more mathematical context. The value of this activity is that it encourages students to use their own problem-solving abilities to pose questions and answer them. Modeling problem solving through a sequence of questions, students are led gradually from concrete to abstract reasoning.

3 Scenario #2: Regions with overlapping circles

A second scenario and a challenging extension question for students is: "Can this formula, or an adaptation of it, be used to describe the number of regions created from overlapping circles rather than cuts?" An identical questioning process to that presented with linear cuts is presented. For example, how many regions are formed from three intersecting circles?

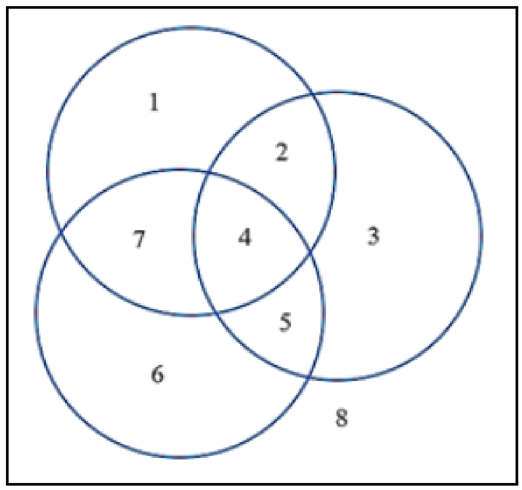

Fig. 6: *A diagram of regions formed from three overlapping circles.*

Explanation: Consider the table in Figure 7 which displays the relationship between the number of circles and the maximum number of regions created including the space outside of the circles. Similar to our last scenario, the maximum number of regions is created by forcing each new circle to intersect all previous circles at two points (instead of one, which is the maximum for a line intersecting another line).

Ask students to compare this table to that of the first scenario. How is this table the same as the table in Figure 4? How is this table different from the table in the original question? Students might notice that both tables start out the same on the first three rows, but then the values begin to differ. They might also notice that the values for the change in the number of regions from the last cut of the pie increase by one, where that same value with the circles increases by two (after the first three rows).

Ask students to speculate why this might be. Students may suggest that it is because a line can only intersect another line once, yet a circle intersects another circle at most two times.

Number of circles (n)	Maximum number of regions created (r_n)	Change in number of regions from last circle
0	1	n/a
1	2	+1
2	4	+2
3	8	+4
4	14	+6
5	22	+8
6	32	+10

Fig. 7: *Maximum number of regions created by n overlapping circles.*

It is also possible to determine recursive and explicit functions for this scenario. The recursive equation is $r_n = r_{n-1} + 2n$ (with $r_0 = 1$) and the explicit equation is $r_n = n(n-1) + 2$, where n represents the number of circles and r_n represents the maximum number of regions formed by these circles. The explicit formula can be used to determine the maximum number of regions created by n overlapping circles. Again, a proof of this can be used to challenge gifted or curious students and be found at: `http://www.90thkilmacudscouts.com/maths/circles_lines_soln.html`

Have students compare the original recursive and explicit equations for both scenarios as in Figure 8. How are they different? How are they the same? Speculate as to why you think these differences/similarities happen.

	Pie Cuts (n represents the number of cuts)	Overlapping Circles (n represents the number of circles)
Recursive Equation	$p_n = p_{n-1} + n$ with $p_0 = 1$	$r_n = r_{n-1} + 2n$
Explicit Equation	$p_n = (\frac{1}{2})(n^2 + n + 2)$	$r_n = n(n-1) + 2$

Fig. 8: *Recursive and explicit equations for each scenario.*

Both scenarios use a sequence of questions to help students solve problems. Pósa's question posing method encourages students to explore a problem using a progression of questions posed by the teacher. Students use previous knowledge to determine a general solution.

4 Conclusion

Pósa's strategy of using a sequence of questions is useful for many problem-solving situations and is designed as a typical strategy that mathematicians might use in their careers. Although I only presented two scenarios in this paper, this strategy can be used with many other scenarios. There are also other strategies that we learned at BSME which I hope to present in the future.

Experiencing Pósa's methods in the BSME program for engaging students in mathematical learning and helping them to feel the excitement of discovery was an invaluable experience for me. I am very fortunate to have been able to study these strategies, especially with Pósa himself. I believe this experience will make me a better teacher, and I am excited to try the strategies soon in my own classroom as well as find new, creative methods that present concepts and structure activities which make discovery natural and memorable.

References

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More information about the BSME Program

The Budapest Semesters in Mathematics Education (BSME) program takes place in Budapest, Hungary, and is available to students for the Fall, Spring, and Summer semesters. In this program, students earn college credits through a variety of classes that focus on teaching mathematics "the Hungarian way." The deadline of the application for the Fall semester is by June 1 of the same calendar year, the Spring semester deadline is by November 1 of the previous calendar year, and the Summer semester deadline is by April 1 of the same calendar year.

The cost of tuition for the Fall 2019/Spring 2020 is $9,995, and the Summer 2020 semester is $4,995. The tuition does not include the costs associated with living in Budapest during the program or transportation to Hungary. Although BSME does not offer financial aid or scholarships, it is often possible for students to receive some or all of the aid granted through their home institution. More information can be found on their website, https://bsmeducation.com/.