Showing Consistency and Comprehension in Solving Quadratic Equations

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Abstract: *The author describes a classroom activity that had students solve quadratic equations in stations. Students were provided with four different solution techniques and selected the 'best' method for each equation. Then they created their own equations. The author discusses revisions to the activity based on her students' misconceptions and research.*

Keywords: *Algebra, quadratic equations*

1 Introduction

1.1 Solving Quadratic Equations

Without question, quadratic equations are a fixture in the Algebra 1 curriculum. Students are asked to solve quadratics on almost every standardized test. Consider the following prompt.

$$\text{Solve: } x^2 + 7x + 12 = 0$$

Significant problem-solving strategies are needed to effectively solve the equation. In the discussion that follows, we present an activity designed to help students solve quadratics efficiently. Students are presented with multiple quadratic equations and use any of the following strategies to solve them: (1) quadratic formula, (2) factoring, (3) square roots, (4) completing the square. Follow-up questions, based on Open Middle resources, ask students to work backwards from zeros, create equivalent quadratic equations in different forms, and construct a quadratic equation with integer zeros.

1.2 Alignment to Standards

The Common Core Standards (2010) covered in this lesson include

- Mathematical Modeling Standard 1: Make sense of problems;
- Mathematical Modeling Standard 5: Persevere in problem solving and Using appropriate tools strategically;
- A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression; and
- A.REI.4 Solve quadratic equations in one variable, with particular focus on part (b) Solve quadratic equations as appropriate to the initial form of the equations by inspection.

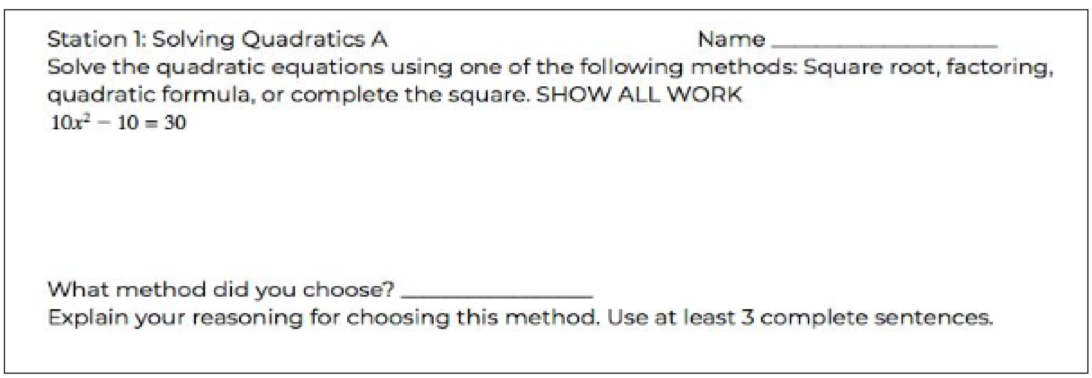
Quadratic equations are most frequently used to model free-fall objects, applications of area, and modeling profit.

1.3 Focus on Multiple Pathways

I did not ask my students to consider real-world applications in the lesson. Rather, my focus was on helping students solve quadratics efficiently and correctly when multiple solution pathways are present. My original lesson plan is available at `https://tinyurl.com/quad-lesson`; an original student handout is available at `https://tinyurl.com/quad-handout`. This lesson was created as stations that students were able to work through at their own pace. Additionally, the first three stations were differentiated, and most students were able to choose what level they wanted to solve. While set up can look very different, I scattered the station tasks in dry erase sleeves around the room. Students recorded their work on separate sheets that they ultimately turned into me. This allowed students to chunk their work and focus on one task at a time. Additionally, I could easily check in with students as they got up to get the next station.

2 Context

The lesson was implemented in an Algebra 1 class with 21 students. The activity took place after End-of-Course (EOC) testing and one week before the end of the school year. We had learned about factoring in March, graphing quadratics and solving using the quadratic formula in April, and had most recently learned about completing the square and using the square root after applying inverse operations to isolate the quadratic term. Students were introduced to each method separately and had very little practice comparing methods or selecting a preferred solution strategy. At this point, I wanted students to see which methods they liked best and find a method that they felt comfortable using consistently (see Figure 1). The first three stations asked students to solve quadratics using methods of their choosing.


Station 1: Solving Quadratics A Name ____________
Solve the quadratic equations using one of the following methods: Square root, factoring, quadratic formula, or complete the square. SHOW ALL WORK
$10x^2 - 10 = 30$

What method did you choose? ____________
Explain your reasoning for choosing this method. Use at least 3 complete sentences.

Fig. 1: *Station 1—Students choose a method.*

3 Student Data and Work

3.1 Student Misconceptions

Most students started at Station 1 or Station 2. The classroom was noisy and busy; students moved among stations while discussing strategies with their peers. In most cases, students worked with a partner to complete the stations. As students worked, many needed help getting started. About 20 minutes into the activity, I noticed a number of misconceptions. For instance, students that used the square root method often forgot to add a $\pm$ in front of their answer. Others who used methods involving factoring or the quadratic formula forgot to manipulate the original quadratic equation so that it's set equal to zero. These misconceptions are illustrated in Figure 2.

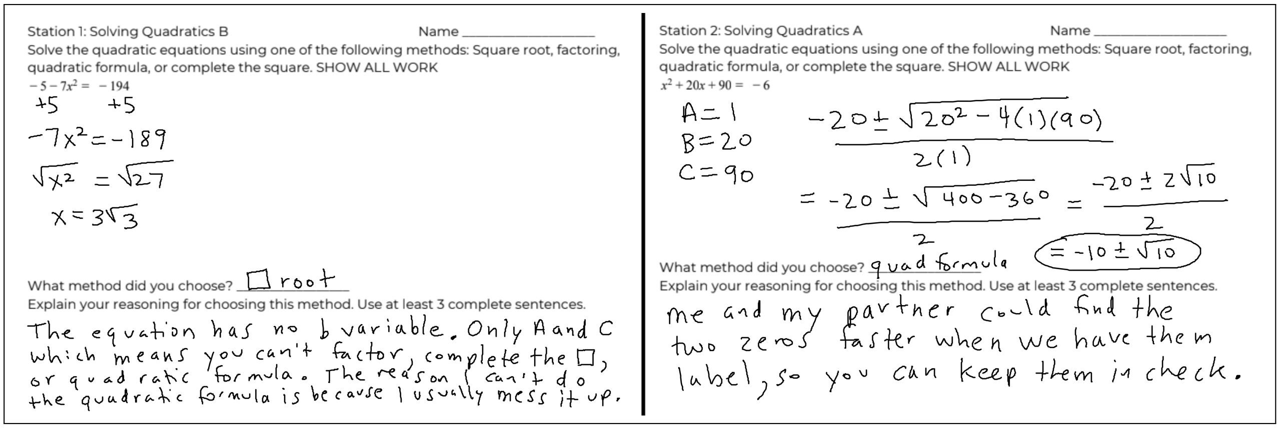


Fig. 2: *Misconceptions with square root method (left, Student A) and quadratic formula (right, Student B).*

At Stations 4 and 5, students were asked to create their own quadratic equations subject to constraints that I provided (e.g., zeros have different values). Students approached the tasks in a number of different ways. In general, they struggled. As the eraser marks in Figure 3, top, suggest, many tried to solve the Station 4 task by guessing coefficients using a quadratic in standard form (rather than starting with a factored form—a far more efficient approach). Figure 3, bottom, illustrates students' tendency to express their final answers as *expressions* rather than as *equations*.

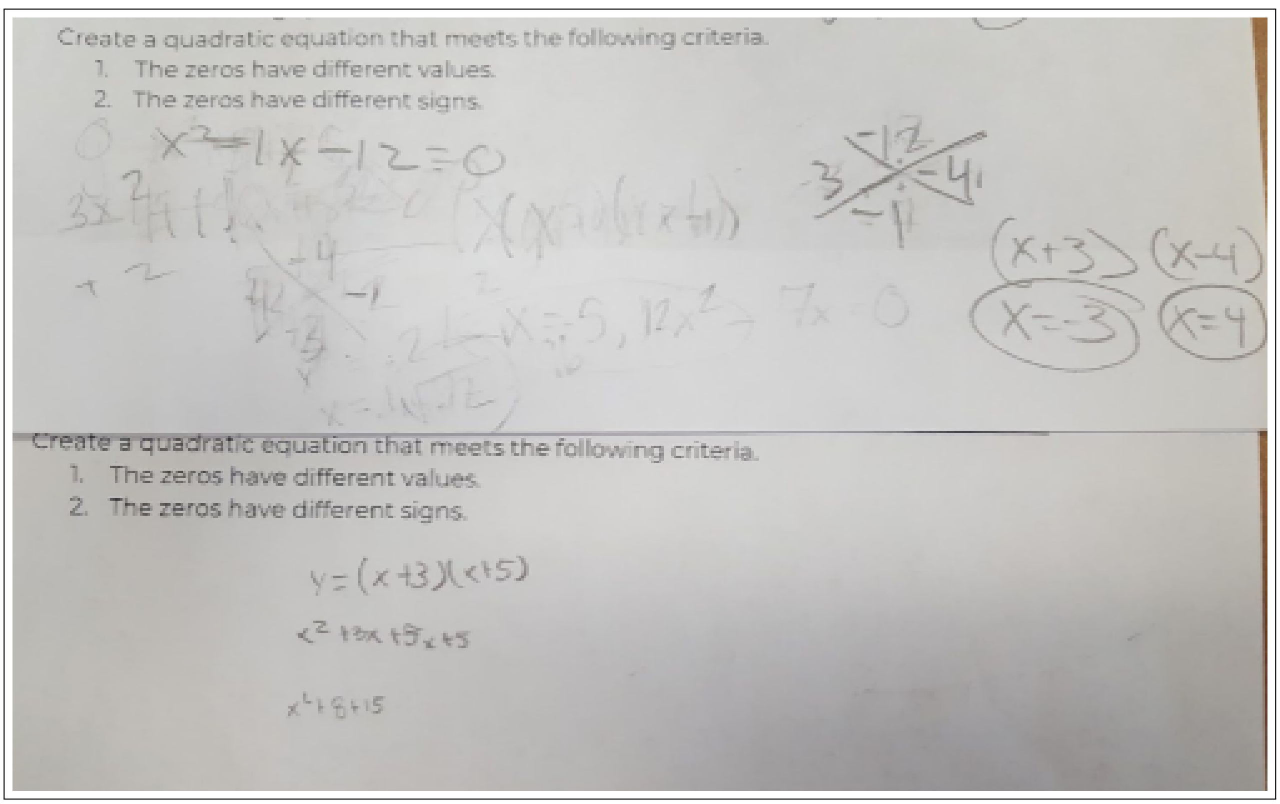


Fig. 3: *Students had a tendency to start with standard form (top) and write final answers as expressions rather than as equations (bottom).*

3.2 Summary of Student Methods

Table 1 lists the problem-solving strategies employed by students for Stations 1 through 3. The frequencies listed in the table suggest techniques that students feel the most confident using. Starting with Station 1, students were most likely to use the square root method and were least likely to use factoring. In Station 2, there was a shift towards factoring and the quadratic formula. The popularity of completing the square decreased across the three tasks. By Station 3, a majority of students used the Quadratic Formula. I find this interesting, especially when considering the numbers for factoring, a sharp uptake going right back down.

	Station 1	Station 2	Station 3
Square Root	7	N/A	4
Factoring	3	7	3
Quadratic Formula	4	8	7
Complete the Square	4	3	1

Table 1: *Analyzing student work by method used for each task.*

4 Literature Review

Quadratic equations have been studied through their many approaches, techniques, and student interactions. In my original activity, students were provided with options to solve each question—using an "open-middle" approach. According to Gael (2016), "Open-middle problems usually have multiple solution pathways, a procedural look, a puzzle-like complexity, and a seemingly simple solution, but often a more complex optimal answer."

However, not all methods are created equally. For example, Bossé (2005) addresses the probability of factorability of quadratics. Most quadratics are not factorable, but most quadratics found in textbooks and teaching resources are factorable (p. 146). This is probably a way to make quadratics more accessible as Didis (2015) stated, "for most students, quadratic equations create challenges in various ways such as difficulties in algebraic procedures" (p. 1138). Students who struggle with the basics of algebra and arithmetic have many barriers to overcome to be successful at quadratics. This is displayed by the quadratic $x^2 - 5x + 6 = 0$; students showed higher success rates with this quadratic than $x^2 + \frac{2}{3}x - 1 = 0$. The reasoning behind this is "it was easy to factor, has rational roots, and students have had more practice and greater procedural abilities in solving these types of equations" (Didis, 2015, p. 1146). In contrast, students struggled more with the second equation because the zeros are irrational.

However, what we are teaching is not necessarily applicable outside of the mathematics classroom, since the probability of factorable quadratics is so low. Bossé continues by advocating for completing the square and the quadratic formula as more applicable ways to solve equations (p. 149), regardless of how much students might struggle. By recommending that students use the quadratic formula and completing the square more frequently, Gael (2016) contends that these problems become accessible by *all* students; "students were not only allowed but also encouraged to explore different representations and tools to use to solve the problem" (Gael, 2016) by allowing them to choose a method with which to solve.

5 Revision Analysis

My original lesson provided students with opportunities to test and grow in their knowledge of solving quadratics. Students participated in mathematical discourse with their peers to identify best methods and check answers (Bostic, 2010). Looking back, however, I realized that the station activities could be strengthened by providing students with more differentiated learning opportunities. Since a number of students needed more practice solving quadratics with different methods, I added two additional stations to a revised version of my lesson. In the two new stations, I ask students to solve the equations using two different methods to check their answers. In addition, I've included quadratics that aren't factorable in an effort to guide students towards the quadratic formula and completing the square, methods work for any quadratic (Bossé, 2005). I've asked students to check their work using an alternative method to address errors associated with the incorrect application of the square root method. These modifications provide students with additional practice before they move on to creating quadratics in the subsequent stations.

In addition to adding stations, I also revised station four by adding multiple criteria and asking students to create an equation that satisfies at least two criteria *(Figure 4)*.

Stations 4: Creating Quadratics Name ____________

Create a quadratic equation that meets at least 2 of the following criteria.

1. The zeros have different values.
2. The zeros have different signs.
3. One zero is a fraction.
4. The coefficient of x^2 is not 1 (a>1)
5. One zero is greater than 10
6. One zero is smaller than 3

Fig. 4: *Revised Station 4 with multiple criteria.*

By providing students with choice, Station 4 provides students with a more differentiated experience, allowing for more student creativity and extensions for those who complete the task early. The "open middle" and "open end" allow students to use techniques and reasoning that are accessible to them (Gael, 2016).

After collecting and deciphering the student work, I realized that there were easier ways to go about organizing this activity for myself. First, I have added a student answer sheet. This will help students stay organized, avoid staples, and I can easily check in with students. I will continue to have the stations scattered around the room to promote movement and to be able to visually check in, but I will only have 5–6 copies of each station in a dry erase sleeve. This also gives students the opportunity to work on the physical station with a dry erase marker and then transfer the work to their answer worksheet.

In order to promote mathematical discourse and vocabulary in my classroom, my colleagues suggested that I add more to the discussion at the end of class. As part of the discussion, students will pick a method they prefer and group themselves accordingly. Then, they will form groups where each person has a different solving preference. Groups will be given an equation and asked to solve it each way. The "professional" must help the other two students solve the equation in their method, but they are not allowed to touch the writing utensil. This provides the students with the opportunity to be the teacher, and give them a chance to express their reasoning for using this method (Bostic, 2010).

To continue to challenge students that have completed before the discussion, I have added extensions that challenge students to create equations that are best solved by the Quadratic Formula, completing the square, and the Square root method. I've de-emphasized factoring based on recommendations by Bossé (2005). Additionally, the new tasks have open middle characteristics in addition to multiple correct answers.

6 Conclusion

My revised activity provides students with more opportunities to reach a level of consistency when solving quadratic equations. The revisions challenge students' procedural know-how while also engaging them conceptually. The added discourse elements encourage students to share their knowledge and verbalize their comfort levels with each of the problem solving strategies. The revisions also give teachers opportunities to discuss quadratics that can't be factored and the need for multiple methods. With the help of my colleagues, we edited a lesson that now has more real-world application, open middle problems, open ended problems, and promotes student discourse with quadratics. What a feat!

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APPENDIX A: Revised Lesson Plan

Lesson Title
Solving Quadratic Equations Stations

Grade Level
9th Grade Regular/Concepts Algebra 1

Lesson Objectives
To allow students to practice the four ways that we have learned to solve quadratics. In practicing this way, students will show consistency in solving with at least one method and problem solve to find the best method for solving each quadratic. Students will practice explaining their reasoning with each quadratic solved.

Ohio Standards, Benchmarks, and Grade Level Indicators
Make sense of problems and persevere in solving them (Mathematical Modeling 1)
Use appropriate tools strategically (Mathematical Modeling 5)
Construct viable arguments and critique the reasoning of others (Mathematical Modeling 3)
A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.
A.REI.4 Solve quadratic equations in one variable.

a. Use the method of completing the square to transform any quadratic equations in x into an equation of the form (x-p)^2=q that has the same solutions.
b. Solve quadratic equations as appropriate to the initial form of the equations by inspection

Materials, Technology, Resources
Student Resources:
Students will need pencils, calculators (scientific), student answer worksheet and access to the stations that will be posted around the room. For #4 - 6, students will be allowed to use Desmos on their computers after 4 attempts algebraically.

Teacher Resources:
Teachers will need the half sheets, dry erase pockets to contain the stations, and a dry erase marker to label each station. Teacher will need an answer key to check student work.

Background Knowledge/Vocabulary
Students should be familiar with factoring, the Quadratic Formula, completing the square, and the square root method. Students should have had practice with each of these methods independently of each other.

Lesson Procedure (Launch, Lesson Core, Closure)
Launch (15 minutes)

1. Show examples of using the quadratic formula, completing the square, and factoring on the board. Use the quadratic x^2+5x+6=0; the zeros are -2 and -3. Leave these examples on the board for students to reference. Be sure to emphasize that the equation must be equal to zero.
2. Explain the procedure for the day. Students should have out a pencil, calculator, and notes as needed. Students will be self-pacing through the stations, and the stations will be collected at

the end of the period. Students will be expected to complete all the stations during class. Explain the directions that are on the sheets and the expectations of solving the quadratic, showing every step and then explaining their work.

Lesson Core (50 minutes)

1. Students will be working either independently or in groups of 2 - 3. Circulate to make sure students are working in correctly sized groups, and students have picked the correct differentiation for them.
2. About every 10 minutes, check in with students. Make sure students have completed the appropriate number of stations, and reemphasize the examples on the board. Continue to check on students individually; check student work and encourage students to work with each other and check each other's work.
3. Students that struggle with stations 1 - 3 should be directed to complete stations 7 & 8 before moving back to stations 4 - 6.

Closure (20 minutes)

1. Have students pause. Check in, how many students used each strategy? What was the most popular strategy? Why did you use that strategy most often? When does each strategy work? Does each strategy work all the time? Some of the time?
2. Group together by similar method
3. Once their grouped, they pick a partner from a different method group
    a. Each partner has to do the other's method, with help from the original "master". For example, Student 1 comes from the "factoring" group and Student 2 comes from the quadratic formula" group. Student 1 would have to do a new problem using quadratic formula and Student 2 would have to solve the same problem using factoring. Make sure students understand that as the student is trying the different methods, the "master" can only assist, not take over. After pairs try a new method, class come back together to discuss challenges and triumphs of each method
4. Explain our next project that students will start tomorrow. Students will need to create two quadratic equations and show how to solve each quadratic in each way. Tell students their homework for tonight is to think about their quadratics.
5. Collect students' station work.

Assessment

Students will be completing a project on creating and solving their own quadratics starting the next day. This project will be graded based on the quadratics they create, and their ability to solve each quadratic using factoring, quadratic formula, completing the square, and graphing. This station activity and project are in preparation for the final exam on quadratics.

Accomodations

The activity has been differentiated for the students. Most students will be able to choose whether they want to do Problem Set A or B. They will also have the option of switching between the sets as they work. Students will be able to work in pairs, and I will be circulating the room to help those as needed. The instructional aide in this class will also be circling to help students as they need it. Students who struggle with stations one through three will be directed to stations seven and eight which will provide additional practice. Once the students finish these, they will complete stations four through six. Additionally, students will be able to use Desmos as needed on stations four through six if they need help.

Extensions

Have students create a quadratic function that:

a) Would be most efficiently solved by the Quadratic Formula (explain why the Formula is the most efficient method)
b) Would be most efficiently solved by completing the square (explain why completing the square is the most efficient method)
c) Would be most efficiently solved by square rooting (explain why square rooting is the most efficient method)

Rubric

	3	2	1
Solving the Quadratic (Station 1 - 3)	Student showed their work and reached the correct answer.	Student showed some work or student reached an incorrect answer with work shown	Student did not show work and did not reach the correct answer
Explaining Their Work (Station 1 - 3)	Student explained their work	Student used less than 3 sentences to explain their work	Student did not explain their work
Creating a Quadratic Given the Criteria	Student fulfilled at least 3 criteria	Student fulfilled one to two criteria	Student did not fulfill the criteria
Creating a Quadratic in Multiple Forms	Student attempted the problem and got the correct answer	Student attempted the problem	The problem was not attempted
Solving the Quadratic in Two Ways (Station 7 - 8)	Student showed their work and reached the correct answer in both strategies	Student showed one strategy or student reached an incorrect answer with work shown	Student did not show work and did not reach the correct answer

APPENDIX B: Revised Student Handout

Station 1: Solving Quadratics A Name ____________________

Solve the quadratic equation using one of the following methods: Square root, factoring, quadratic formula, or complete the square. SHOW ALL WORK

$10x^2 - 10 = 30$

What method did you choose? _________________

Explain your reasoning for choosing this method. Use at least 3 complete sentences.

Station 1: Solving Quadratics B Name ____________________

Solve the quadratic equation using one of the following methods: Square root, factoring, quadratic formula, or complete the square. SHOW ALL WORK

$-5 - 7x^2 = -194$

What method did you choose? _________________

Explain your reasoning for choosing this method. Use at least 3 complete sentences.

Station 2: Solving Quadratics A Name ___________________

Solve the quadratic equation using one of the following methods: Square root, factoring, quadratic formula, or complete the square. SHOW ALL WORK

$x^2 + 20x + 90 = -6$

What method did you choose? ________________

Explain your reasoning for choosing this method. Use at least 3 complete sentences.

Station 2: Solving Quadratics B Name ___________________

Solve the quadratic equation using one of the following methods: Square root, factoring, quadratic formula, or complete the square. SHOW ALL WORK

$-9x^2 + 8x + 8 = -10x^2 - 7$

What method did you choose? ________________

Explain your reasoning for choosing this method. Use at least 3 complete sentences.

Station 3: Solving Quadratics A Name ____________________

Solve the quadratic equation using one of the following methods: Square root, factoring, quadratic formula, or complete the square. SHOW ALL WORK

$2x^2 + 9x + 21 = 11$

What method did you choose? _________________

Explain your reasoning for choosing this method. Use at least 3 complete sentences.

Station 3: Solving Quadratics B Name ____________________

Solve the quadratic equation using one of the following methods: Square root, factoring, quadratic formula, or complete the square. SHOW ALL WORK

$11x^2 - 21 - 4x = -4x$

What method did you choose? _________________

Explain your reasoning for choosing this method. Use at least 3 complete sentences.

Stations 4: Creating Quadratics Name ____________________

Create a quadratic equation that meets at least 2 of the following criteria.

1. The zeros have different values.
2. The zeros have different signs.
3. One zero is a fraction.
4. The coefficient of x^2 is not 1 (a>1)
5. One zero is greater than 10
6. One zero is smaller than 3

Station 5: Creating Quadratics Name ____________________

Use the digits 0 to 9 at most one time each, fill in the boxes to create three equations that produce the exact same parabola.

$$y = (x + \square)^2 - \square$$

$$y = (x + \square)(x + \square)$$

$$y = x^2 + \square x + \square$$

Station 6: Creating Quadratics Name ____________________

Use the digits 0 to 9 at most one time each, fill in the boxes so that the solutions are integers. Note that you can use the 05 to make a 5.

$$\square x^2 + \square\square x + \square\square = 0$$

Station 7: Solving Quadratics Name ____________________

Solve the quadratic equations using TWO of the following methods: Square root, factoring, quadratic formula, or complete the square. SHOW ALL WORK

$x^2 - 7x = 40$

Station 8: Solving Quadratics Name ____________________

Solve the quadratic equations using TWO of the following methods: Square root, factoring, quadratic formula, or complete the square. SHOW ALL WORK

$2x^2 + 17x - 29 = -20$

APPENDIX C: Station Recording Sheet

Solving Quadratic Equations Stations Student Worksheet Name ________________

Directions: Record work and explanations below. Be sure to mark whether you do A or B for Stations 1 - 3.

Station Work	**Explanation**
Station 1: A or B	
Station 2: A or B	
Station 3: A or B	

Station Work
<u>Station 4</u>
<u>Station 5</u>
<u>Station 6</u>
<u>Station 7</u>
<u>Station 8</u>