Looking for Pythagoras

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Abstract: *The author discusses the process and revision of a lesson implemented in a 7th-grade classroom. Students explored triangles and the areas associated with their side lengths in order to discover the Pythagorean theorem. The lesson was then revised to use tools such as patty paper and GeoGebra software.*

Keywords: *geometry, algebra, measurement, hands-on activity*

1 Introduction

In the paragraphs that follow, I describe my 7th graders' exploration of the Pythagorean theorem in our pre-algebra class. In the original activity, students found the areas of squares created by the side lengths of a right triangle, an acute triangle, and an obtuse triangle. They looked for patterns and relationships between these areas. After some initial discussion in small groups and a whole class setting, I asked them to create their own triangles and test their conjectures. The main objectives of the lesson were as follows:

- **Measurement**: Students engage in explorations of different types of triangles as they calculate areas of squares formed from the sides of each triangle;
- **Reasoning and Proof**: Students find possible relationships between different types of triangles and the areas of squares formed from the triangle's sides. They test their conjectures with additional examples of their own creation.
- **Problem Solving**: Students discover the Pythagorean Theorem through their explorations. The exploration process fosters development of students' problem solving skills and mathematical reasoning.

This lesson was taught during Spring 2019 and was revised with 12 other practicing middle and secondary-level mathematics teachers as part of EDT 566, an Advanced Teaching Methods course offered by Miami University during the following summer. Below, I discuss the student activity (and associated student work) in greater depth and describe a number of research-informed revisions that I made to the lesson with the assistance of my teaching colleagues.

2 Lesson Context

2.1 Previous Student Learning

Immediately prior to the Pythagorean Theorem lesson, my students had just learned about square roots through an exploration of the relationship between the area of a square and its side length. Once students realized that the side length squared equalled the square's area, we worked backwards and established that the square root of the square's area is equal to its side length. Students practiced this with many squares, including examples for which the calculations of the side length could only be determined with the square root (i.e., side lengths weren't integer lengths and couldn't be determined accurately with grid paper). In addition, my students had learned how to determine the lengths of line segments by constructing a square from each segment on grid paper, finding the square's area (by counting unit squares), and then calculating the area's square root with a calculator. Eventually students began to ask if there was any easier way to find lengths other than counting unit squares and taking the square root. This curiosity sparked their interest in the Pythagorean Theorem lesson.

2.2 Importance of the Theorem

Knowledge of the Pythagorean Theorem is essential for my students—First, a formal statement of the theorem simplifies the process my students have been using to calculate unknown side lengths of squares and triangles while providing them with a generalized method that works with or without grid paper. Secondly, students will encounter the Pythagorean Theorem repeatedly throughout subsequent coursework—in geometry and algebra courses and as they study trigonometry and calculus in high school and beyond. Moreover, many real-world applications such as carpentry, architecture, and astronomy use the theorem. Students apply the Pythagorean Theorem to calculate the length of objects such as roof trusses (and the radius of the earth!) indirectly and to find the shortest path to a certain location. The activity aligns to Common Core Standard 8.G.B.7—*Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems in two and three dimensions.*

3 Student Data

Before beginning this lesson, only one or two of the students had heard of the Pythagorean theorem. They were not made aware of its existence nor was the lesson introduced in any way that would lead students to believe we were looking for a formula that would solve all of our square counting problems. Rather than introduce the formula and provide practice, I wanted students to explore triangles to see if they could discover it on their own. I asked the few students who did know the formula to keep it a secret until the end of our activity and to continue looking for other patterns. They happily obliged. :)

I distributed a structured inquiry handout to my students (a digital copy is available at `https://tinyurl.com/pythagoras-worksheet`). In small groups, students began exploring—analyzing triangles, counting squares, calculating square roots with calculators, and finding areas. They shared ideas with others in their groups and other groups throughout the classroom. As discussions began, some ideas were mathematically correct and others were not. All ideas were welcomed—after all, there is no such thing as "incorrect brainstorming," and all conjectures provided students with questions to be further explored. For instance, several groups considered adding side lengths of the two smaller squares to find the side length of the largest square (i.e., $a + b = c$). Work supporting this conjecture is illustrated in Figure 1 and was typical of students in my classroom.

Some of the calculation methods highlighted in Figure 1 may be unfamiliar to secondary-level teachers. Note, in particular, the method that the student used to find the area of the square with side length c. Since the student doesn't know the Pythagorean Theorem, she encloses the square inside a larger (red) square formed by drawing 4 congruent triangles along the outside of the original square. This large, red square has side length 7 units (and thus area of 49 square units). The student obtains the side length through simple counting and the area by multiplying 7 units $\times$ 7 units. The area of the square with side length c is determined by subtracting the areas of the 4 congruent right triangles (6 square units each) from 49 square units ($49 - 4 \cdot 6 = 25$).

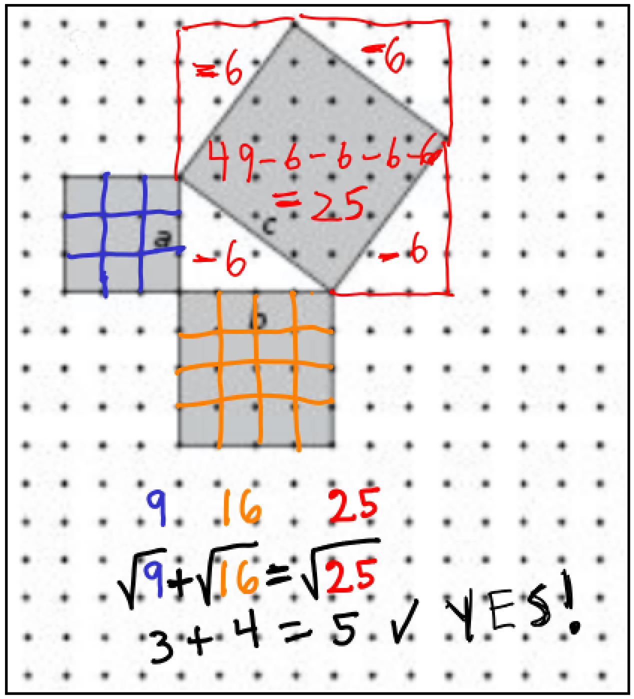

Fig. 1: *Based on their work with a 3-4-5 right triangle, some students incorrectly generalized that $a + b = c$ for all triangles with side lengths a, b and c.*

Soon, students discovered that their initial conjecture didn't hold for all triangles (or even for all right triangles). This is realization is suggested in the work shown in Figure 2.

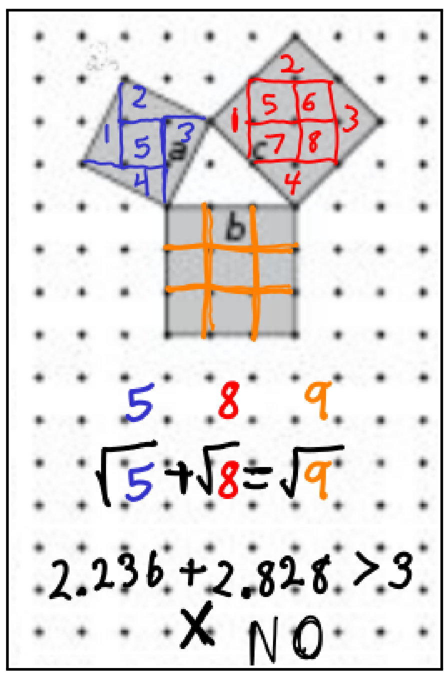

Fig. 2: *Work illustrating that $a + b = c$ does not hold for all triangles with side lengths a, b and c.*

As students progressed through the worksheet and discussion between students continued, mathematically correct relationships emerged. Figure 3 highlights a table of values from Item 2 of the worksheet completed by another student. The chart was designed to help students compare relationships among acute, right, and obtuse triangles.

Type of Triangle	Area of Square on Side a (units2)	Area of Square on Side b (units2)	Area of Square on Side c (units2)	Length of Side a (units)	Length of Side b (units)	Length of Side c (units)
1. Right	9	16	25	3	4	5
2. Right	10	9	1	3.16	3	1
3. Acute	5	9	8	2.236	3	2.828
4. Acute	13	16	13	3.60	4	3.6
5. Obtuse	13	37	68	3.605	6.082	8.246
6. Obtuse	8	25	40	2.82	5	6.32

Fig. 3: *Table of values comparing side lengths and areas associated with 6 different triangles—2 acute, 2 right, and two obtuse*

The work in Figure 3 was typical of other students' work in the class. After completing the table, many successfully concluded the following: (i) $a^2 + b^2 = c^2$ for right triangles, (ii) $a^2 + b^2 > c^2$ for acute triangles, and (iii) $a^2 + b^2 < c^2$ for obtuse triangles (see Figure 4). Most students were able to articulate these ideas out loud, although few wrote these conclusions formally as inequalities until we discussed them as a whole class.

For each triangle, look for a relationship among the areas of the three squares on the sides. Make a conjecture about the areas of the squares drawn on the sides of a triangle and type of triangle.

if A and b get added together in the obtuse the small sqaures are going to be less if get larr acute is the smalls sqaurs are greater then c and right is $a^2+b^2=c^2$

Test your conjecture by drawing another triangle.

$a^2+b^2=c^2$

$a^2+b^2>c^2$

$a^2\ b^2<c^2$

$1+1<2$

Fig. 4: *Relationships between lengths of sides for right, acute, and obtuse triangles.*

3.1 Lesson Challenges and Revision Opportunities

3.1.1 Student Difficulties

One of the most common challenges students had as they completed the lesson involved the creation of original examples to test their conjectures. Many middle school students had difficulty creating triangles and squares that were easily "countable." As Figure 5 illustrates, students had difficulty drawing straight lines and polygons on dot paper. The inability to construct accurate examples led many students to form faulty conclusions. Although the work in Figure 5 includes area calculations and manipulations, it's unlikely that accurate conclusions could be made from the shapes drawn on the page.

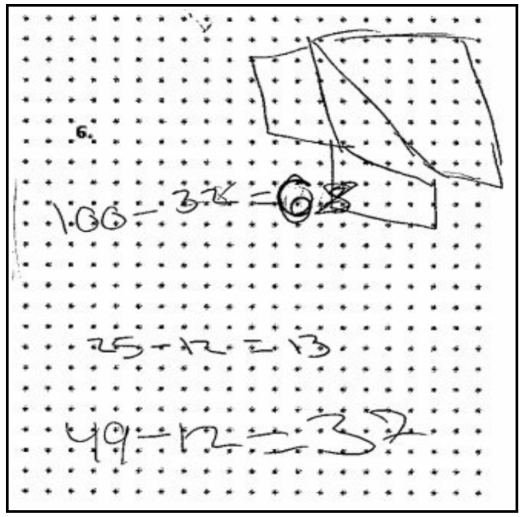

Fig. 5: *Students struggled to draw useful examples to test their conjectures.*

3.1.2 Assessment of Student Learning

It is difficult to score students in this type of exploration. The main purpose of this activity was to allow students to explore, communicate, and draw conclusions in a low pressure environment. There were many group and class discussions along the way and some students went back and corrected mistakes and misconceptions from earlier in the lesson. From analyzing student data, I was able to summarize some of the most common errors. There is some overlap between understanding the Pythagorean theorem as $a^2 + b^2 = c^2$ and the misconception that $a, b,$ and c had to be whole numbers. Some students also thought that you could add the side lengths is some way that was meaningful. Figure 6 summarizes these findings.

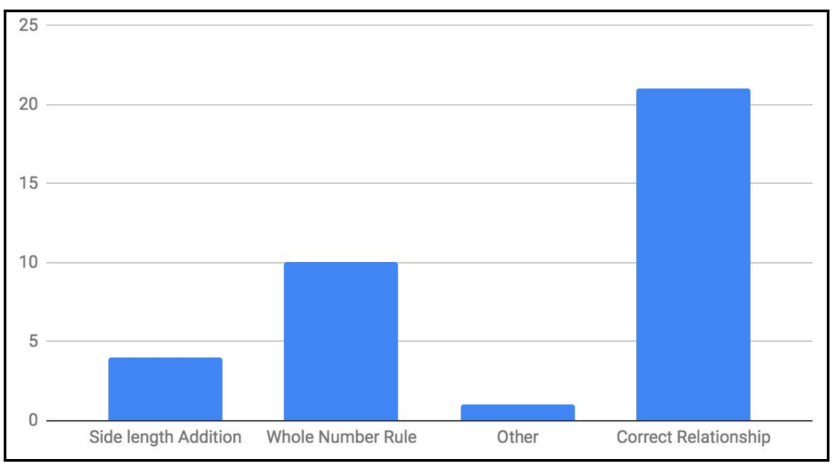

Fig. 6: *Summary of Student Conclusions.*

4 Revision Analysis and Review of the Literature

4.1 Enhanced Discovery Learning

Looking back on the lesson after working with colleagues and researching the topic in a graduate school classroom, I've made several adjustments to the lesson. A revised plan is available at `https://tinyurl.com/lesson-revision`. A revised student handout is available at `https://`

tinyurl.com/revised-handout. Many of the revisions made were to increase the accessibility of the relationships that students were being asked to find. As previously noted, many students had trouble finding correct areas and thus were lead to incorrect conclusions. In "The Perils and Promises of Discovery Learning," Marzano (2011) describes the difference between unassisted and enhanced discovery learning. Discovery learning is instruction in which students interact, explore, and manipulate objects to answer questions and discover relationships. Marzano argues that to do so with no assistance will result in inadequate learning of content.

4.2 Patty Paper and GeoGebra

In the revisions for this lesson, I strove to construct teaching materials that would encourage my students to engage in enhanced discovery learning. By modeling types of triangles and then asking students to make their own, too many struggled to recognize patterns from an insufficient number of examples—moreover, many others lacked the fine motor skills to create their own. I wanted my revisions to retain the discovery-oriented flavor of the original activity, yet provide my students with better support for creating their own examples to test conjectures. With the help of colleagues and the concept of enhanced discovery learning, I decided to incorporate Dynamic Geometry Software (e.g., GeoGebra) and patty paper in the revised lesson. Patty paper allows students to trace existing squares and line them up on the dot paper in a way that is easily countable. Geogebra has many features that allow students to find area as well as see multiple examples in a short amount of time.

4.3 Mathematics History

The other revision created was the incorporation of math history into the lesson. In the article "The 2500-Year-Old Pythagorean Theorem," Veljan (2000) provides a brief history of Pythagoras along with an interesting application of the theorem that I've incorporated into my revision. Vejan provides the following example (p. 261):

> Here is a nice informal interpretation of the theorem: A pizza shop makes three sizes of pizzas; their diameters are the sides of a right triangle. Then the big pizza is equal to the sum of the two smaller pizzas.

This situation is incorporated in the extension portion of the revised lesson and provides students with a relatable context for exploring the Pythagorean Theorem.

References

Marzano, R. (September 2011). The Perils and Promises of Discovery Learning. *Educational Leadership*, 86–87.

Veljan, D. (2000). The 2500 Year-Old Pythagorean Theorem. *Mathematics Magazine 73*(4), 259–272. Available on-line at https://www.jstor.org/stable/2690973

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