The Challenges of Finding Area

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Abstract: *The author explores a rich task that engaged her high school geometry students in procedural and conceptual understandings of area while discussing various student misconceptions that the task uncovered. The author discusses revision ideas generated by research and interactions with colleagues during an advanced methods course for practicing teachers.*

Keywords: *area, geometry, rich tasks*

1 Introduction

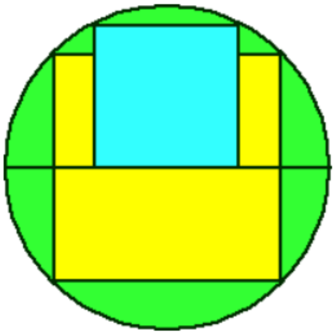
One of the best things teachers can do to enhance the quality of their instruction is to engage in peer-review of lesson plans with trusted colleagues. This past summer, I did exactly that. As a student in the Masters of Arts in Teaching (MAT) Program at Miami University, I enrolled in *Mathematics Misconception Diagnosis & Remediation* (EDT 566), a three-week graduate course focusing on student misconceptions and lesson plan revision. Each of the 12 secondary-level mathematics teachers enrolled in the course collected video and student work samples from a rich-task that they posed during the previous Spring as part of a mini action-research project. During the summer class, we watched video recordings of our peers' instruction, analyzed sample student work, and suggested revision ideas to help remediate misconceptions that we uncovered in the artifacts. By the end of the graduate course, each teacher had a collection of peer-reviewed lessons for use during the upcoming school year. In the following paper, I share my original and revised lessons along with constructive feedback and research provided by my peers.

2 Context

2.1 My Students, Lesson, and Task

For my project, I decided to pose a task from the NRICH website (`https://nrich.maths.org/`) with students in three sections of College Prep (CP) Geometry students (78 students in total). The task as it originally appears at the NRICH website is provided in Figure 1 and is part of the lesson provided at the following link: `https://tinyurl.com/y6smje5n`.

In the diagram below, the blue square is inscribed in the semicircle, and the yellow square is inscribed in the circle.



The blue square has an area of 40cm^2.

Can you find the area of the yellow square?

Fig. 1: *The Semi-Detached Task (`https://nrich.maths.org/1863`).*

The CP track is intended for "average" students and can be challenging to teach. CP classes meet daily for 49 minutes. This doesn't allow much time for exploration, problem solving, or inquiry-based learning. Moreover, the ability levels and ages of the CP students vary significantly. Of the 78 students, 12 are "advanced" freshmen (8 of whom are in the same section of the course). The variety of students in the CP classes definitely leads to an interesting learning dynamic, with sophomores looking up to first-year students as "math gods." Because the first-year students weren't accustomed to exploring rich tasks, I hoped that the NRICH tasks might provide my sophomores with an opportunity to contribute solution ideas that didn't occur to my freshman. In this way, the task might encourage newfound confidence in my sophomores.

2.2 Timing and Logistics

I posed the task to my students shortly after the administration of our end-of-course (EOC) state tests. As such, my students initially pushed back on the idea of solving a non-routine problem. They knew we only had a month of school left, and they were brain-fried and perhaps a bit scarred from the state tests. To encourage my students to engage in the NRICH task, I reminded them that their work would not be graded for accuracy (I was collecting their work for a class I was taking in a few weeks) and that they already knew everything they needed to solve the task correctly—namely, areas of circles, squares, and the Pythagorean Theorem (although I didn't provide this level of specifics to the students). This news perked them up. Our last unit before the state tests was circles, where students saw many inscribed and circumscribed polygons, while the unit before that was right triangle trigonometry, so my students were well-prepared (even if they didn't know it!).

2.3 Standards

The lesson aligns well with our school initiative of promoting problem-solving and higher order thinking, incorporating the first three *Standards for Mathematical Practice*—namely, (1) Making sense of problems, (2) Reasoning quantitatively, and (3) Critiquing the reasoning of others (*Ohio Learning Standards*, 2017, p.4–5). In addition, the lesson is aligned with the following *Ohio Mathematical Standards*: solving problems involving right triangles (G.SRT.8) and identifying relationships among circle properties to solve problems (G.C.2) (*Ohio Learning Standards*, 2017, p.81–82).

3 Posing the Problem

3.1 Bell Ringer

I began class as I typically do, with a bell ringer to get students working and to ensure that they were well-prepared for the day's lesson (see `https://tinyurl.com/y3jh2plu`). After a few minutes, I moved students into pre-assigned ability-based groups and assigned them variations of the original NRICH task (see Tasks 1–4 here: `https://tinyurl.com/yyjt8rr6`).

3.2 Differentiating the Task

Each assigned task included the image from the original NRICH problem (see Fig. 1) consisting of three shapes—(1) a large green circle, (2) a yellow square inscribed in the green circle, and (3) a blue square inscribed in the semi-circle of the green circle. I told students that the blue square had an area of 40 square units. From there, each group was asked to determine areas of different regions. Some were asked to find the area of the yellow square (i.e., Task 1); some, the green circle (i.e., Task 2); others had to find the area of a circle inscribed in the blue square (i.e., Task 3). The rest had to find as many areas as they could (i.e., Task 4). I split Tasks 1 and 2 equally among my "middle" students. I assigned Task 3 to students who tended to struggle and Task 4 to "advanced" learners.

I told the students that we were doing a think-pair-share activity. They would have five minutes to think on their own then would work with their group. The last 10 minutes of class would be reserved for whole group sharing and discussion. I informed students that I would grade their work on process rather than accuracy. As soon as I let them go, I saw dead stares everywhere.

MY MIND : Uh-oh. Did I not explain something? Are they just tired? Oh Lord, please let this work.

Finally, the students started working together, and I quickly discovered why there were so many dead stares.

LUCY : So which one is the blue square?
BOB : I'm pretty sure it's the small one . . . I think.
LUCY : Wait, it has to be because it says the blue square is inscribed in the semi-circle.

Unfortunately, I don't have access to color printing at my school. Although I realized color would be helpful, I assumed that the written descriptions of the shapes would suffice. Clearly, I assumed too much. To clear up confusion, I projected a color version of the task on our smartboard.

4 Student Work

The tasks generated interesting math talk. All students, regardless of level, had to figure out how to use the area of the blue square to determine other areas. Most realized fairly quickly that they could find the side length of the blue square; however, the conversation related to this surprised me.

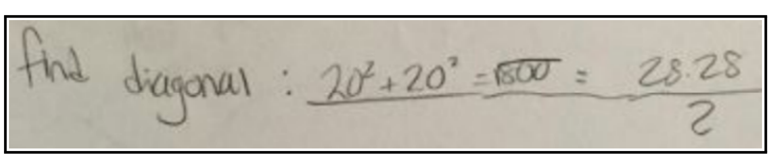


Fig. 2: *Using a side length of 20.*

In Figure 2, notice that the student finds the length of the diagonal of the blue square using the Pythagorean Theorem. While the student's use of right triangle properties excited me, the numbers in his calculations—namely, 20 and 20—did not. Many students made the same error—mistakenly believing that a side length of 20 units creates an area of 40 square units. Luckily, the following conversation was also commonplace:

DAVE : Okay, so if the area is 40 and it's a square, which means the side lengths are the same, then it must be 20!
KATE : Yes, exactly! Okay, so next we need to find . . .
DYLAN : Wait guys, that doesn't make any sense. You have to multiply for area; 20 times 20 is definitely not 40. We need to take the square root of 40.
KATE : Oh my goodness, how did I not see that? Right, because the square root of 40 times itself cancels out the square root.

Although I was concerned that so many students struggled with this concept, I was happy when Dave's group figured out the side length without my help. They troubleshot successfully, and that's all I could ask for (and more).

Unfortunately, side length continued to cause issues for other groups. For instance, many students confused perimeter and area, or forgot that both side lengths of a square are the same—assuming that side lengths were four and ten.

SHARON : All right, the area of the blue square is 40, so each side must be 10 because 10 plus 10 plus 10 plus 10 is 40.
EVAN : Huh?
MADISON : Wait, how does that work?
EVAN : Sharon, that's not right. It asked for area, so it needs to be two numbers that *multiply* to give you 40. So, square root of 40.
ME : Why is it square root of 40?
SHARON : Oh, because it has to be the same and square root 40 times square root 40 is 40.

These students did not need my help, which was encouraging, but it was interesting to see their confusion with area and to listen to their conversations.

Next came the most difficult part—figuring out where to go next. For students completing Task 3 (the lowest level), next steps were relatively straightforward. After sketching a circle within the blue square, many noticed that dividing side length of the square in half gave the radius of the inscribed circle. Plugging this value into the circle area formula, $A = \pi \cdot r^2$, yielded a correct solution, as shown in Fig. 3.

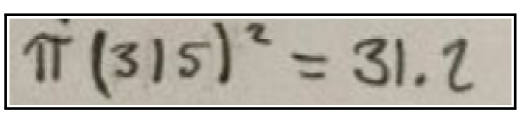


Fig. 3: *Using half the side length as radius to find the area.*

TJ : Guys, if we can find the radius, then we can find the area.
MAX : Well, if we have a side length and it's a square and the diameter is the same size as the square, then the diameter must be 6.32 (i.e., $\sqrt{40}$).
BEN : Okay, so 6.32 squared times π.
ME : Why 6.32 squared?
TJ : No guys, that's the diameter. Area uses radius. I just said that!

MAX : So 6.32 divided by two is what?
TJ : 3.15, and then we square it and multiply it by π to get 31.2.
MAX : Are we right?
ME : I don't know, you tell me.
MAX : I think we are right! Guys, look at us, we finished before everyone else. We're awesome!
ME : Good job! Now what if I inscribed a circle in the yellow square? Do you think you can find the area then?

I was amazed at how fast my "struggling" students completed the task. Perhaps more importantly, the task provided a confidence boost. Not only did my students find the correct answer, they finished before everyone else. This allowed me to challenge them with a task requiring the same work of the more "advanced" groups. I was not expecting this to happen—but I was delighted that it did. I could now keep all of my students on the "same page."

For other groups, it was not that easy. Some students had no idea where to go next. Others realized they needed to find the radius of the green circle before they could find any other information. Many made unsubstantiated assumptions based on the way that the figure looked, despite my daily admonishments not to do so.

Figure 4 represents the most common misconception across all three classes. Many students assumed that the yellow square "looked like" it was double the area of the blue square. Yet, when I asked students how they knew this to be true, they had no response. The assumption is understandable, since the yellow square was inscribed in the whole circle and the blue square was inscribed in *half of the same circle*. Unfortunately, the students' assumption was false.

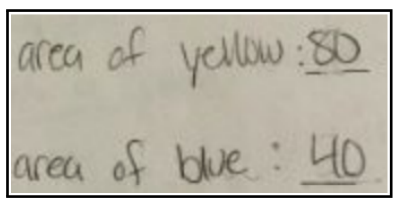


Fig. 4: *Assuming the yellow square was double.*

Another common misconception involved the use of special right triangles. In Figure 5 below, a group of students tried to find the diameter of the green circle by wrongly assuming that the triangles they created were 30-60-90 special right triangles. Interestingly enough, their answer was very close to the correct one since the actual triangle is nearly 30°-60°-90°. This led me to consider changing the dimensions that I originally provided students.

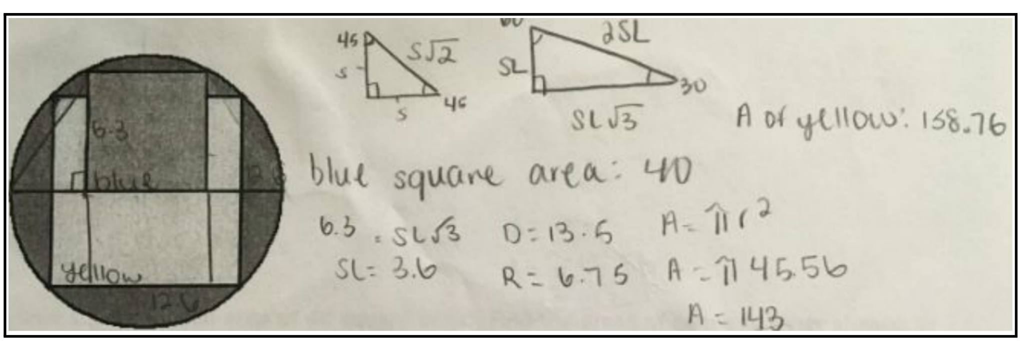


Fig. 5: *Falsely assuming* 30°-60°-90° *triangles.*

Finally, after some group deliberation, students realized that there was enough information to determine the radius as long as they drew the radius inside the blue square. Figure 6 depicts work from a group who figured this out and then tried some extensions of pentagons inscribed in the circle.

BRYCE : We think we got it!
ME : Okay, show me!
BRYCE : Okay, so we drew a line down the middle and realized that it cut the blue square in half. From there, we drew a line from the midpoint to the corner of the blue square to create a 30°-60°-90° triangle.
ME : Wait, how did you know it was a 30°-60°-90° triangle?
ADAM : We used a protractor!
ME : Um, okay, but how could you check to make sure?
BRYCE : Pythagorean theorem.
ME : Okay, can you do that then?
BRYCE : We already did! So, then we found the length of the radius to be 7.04 and plugged it into the circle formula to get the area of the green circle. And then for the yellow square, we doubled the radius to get 14.08 and then used a 45°-45°-90° triangle to find the side length and then we got the area to be 100.

Honestly, my favorite part of that conversation was that they used a protractor, but I was glad that my students were recognizing how important it is to confirm and check their work.

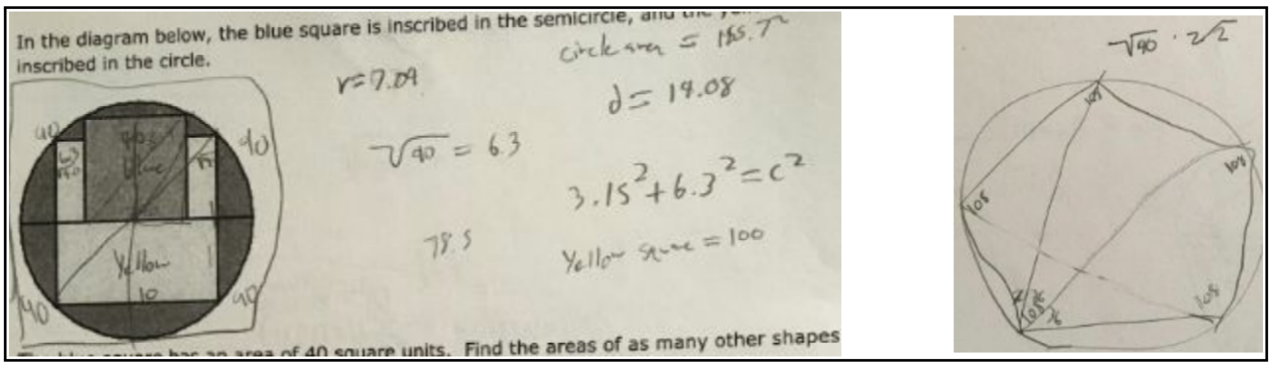


Fig. 6: *Drawing the radius inside the blue square and the pentagon extension.*

We finished the class with students sharing ideas across groups. I assumed a role as a "guide on the side" and let my students challenge each other with their inquiries. It was rewarding to observe the level of discourse happening in the classroom. When several groups insisted that the area of the yellow square was double that of the blue square, one student went up to the front of the room and showed them how that was not possible, cutting the blue square into pieces and fitting it into the yellow square. Table 2 shows the different strategies that students employed for Tasks 1, 2, and 4. Table 3 shows strategies for Task 3. For reference, Table 1 shows a description of each task.

Table 1: Description of Tasks 1–4.

	Task 1	**Task 2**	**Task 3**	**Task 4**
Description of Task	Find the area of the yellow square	Find the area of the green circle	Find the area of a circle inscribed in the blue square	Find the area of as many shapes as possible

Table 2: Frequency of Student Strategies for Tasks 1, 2, and 4.

	Drew radius through blue square	Assumed yellow square was double blue square	Used special right triangles	Assumed blue square was in ratio to yellow square other than 1:2	Other or no clear response
n	30	16	4	1	7

Table 3: Frequency of Student Strategies for Task 3.

	Divided side length by 2 to find radius, then solved for area	Used special right triangles
n	14	1

Ultimately, just over half of my students generated correct answers (despite the fact that I indicated I would only be assessing their work for process). Of the 41 students who calculated the area of the yellow square (i.e., Task 1), 12 did so correctly (i.e., 29%). Of the 40 students who calculated the area of the green circle (i.e. Task 2), 24 did so correctly (i.e., 60% correct). Of the 15 students who calculated the area of the blue circle (i.e., Task 3), all did so correctly.

5 Revisions and Literature Review

Discussions with colleagues during our summer class led to a number of insightful revision ideas. While we discussed many enhancements to the original lesson, two overarching themes became clear: (1) I need to provide more options for my students to test their conjectures, and (2) the lesson would benefit from the addition of an extra day for extension activities. These revision ideas were inspired from articles that we read prior to our revision brainstorm.

5.1 Model Problem Solving for Your Students

Bostic and Jacobbe (2010) discuss how to promote problem-solving discourse in the classroom. They note that problem-solving discourse is "student-centered, problem-focused dialogue among small groups or a whole class" (p. 34). This was an important goal for me when teaching this lesson. I wanted my students to converse with each other rather than relying on me for answers or ideas. In the conclusion of their study, Bostic and Jacobbe note several features of instruction that are essential for promoting that discourse. First, the teacher needs to be a "problem solver" (p. 35). In other words, teachers need to model problem solving for their students—sitting at a desk like their students and engaging in tasks. This allows for more authentic conversation with students. While this was a little difficult for me while video recording, I worked on several tasks at the same time as my students, sharing ideas with students and asking for help and advice. I heard some interesting conversations through this method.

5.2 Provide Flexible Work Time for Students

Second, Bostic and Jacobbe (2010) note that teachers must support the learning needs of their students. They suggest allowing students to pick where they want to sit and asking them how they want to learn mathematics. This led to a revision suggested by a colleague. She agreed that it was good to place students in ability-based groups, but from there, allow them to branch off and work alone if needed or even sit somewhere else. As a revision (see Revised Lesson Plan here:

https://tinyurl.com/yxw8naa2), I plan to allow students to use more time to work alone, but require them to check in with their group every 10 minutes to share ideas.

5.3 Provide Ample Time for Students to Share Ideas and Test Conjectures

Third, Bostic and Jacobbe (2010) suggest that teachers should encourage their students to share ideas every day. This idea led to my revision of adding more options for students to test their conjectures. Many students were stuck on whether the area of the blue square was really half the area of the yellow square or not, but it was hard for me to get them to convince themselves that it was not because there really was not any way for them to test this hypothesis. So, I have added a Geogebra applet for students to explore conjectures as well as more resources like scissors and larger versions of the picture. I also removed the color from the task and changed the title to avoid misleading students. This revision will also help remediate students who thought they found special right triangles (see Revised Student Handout here: https://tinyurl.com/y2b6g3zt).

5.4 Provide Real-World Application / Context

5.4.1 Pegs and Holes

Fourth, the authors suggest that teachers use problems that are relevant, engaging, and have multiple entry points (p. 35). Looking back at this lesson, I really gave no context to my students as to why they needed to solve this. Instead, I just told them to do it. My students were great and did what I asked, but for many of them, they probably did not see any value in it. Cue the revision of turning this lesson into two days. After reading another article, my colleagues and I determined that it would be helpful to create a day two lesson with a real-life application. One article focuses on a problem that explores areas of circles and squares in a real-world context—namely, by exploring which fits better: a square peg in a round hole or a round peg in a square hole (Pritchard, 2010).

5.4.2 Carnival Games

In terms of revision, a colleague of mine came up with the idea of a carnival game. Suppose you are going on a date and you really want to win your date a teddy bear. There are two games that you could play: one where you have to throw a circular ring around a square peg and another game where you have to throw a square ring around a circular peg. Which game do you have a better chance of winning? With this, students will need to calculate the area of the square and round pegs and the area of the circular and square rings and then choose the game that actually has a bigger ratio of peg to ring. This is a great second day activity for students to really explore more with area. Pritchard goes on to discuss further extensions of this problem including nesting (such as a square inside a circle inside a square) and packing small square pegs inside the empty area (Pritchard, 2010). These will be added as extensions to the revised lesson (see Day Two Extension here: https://tinyurl.com/y3t4pdbx).

6 Final Thoughts

Although I knew this lesson could use revision, I would have never been able to come up with the majority of these ideas without my colleagues. I want to encourage any teacher who has the opportunity to reflect and receive constructive feedback to do so because it is worth it. I am excited to introduce this new and revised lesson in my classroom and hopefully explore some new conjectures from there.

References

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