Problem Solving the Hungarian Way (Part 2)

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Abstract: *Lajos Pósa (pronounced Posha) is one of the most prominent mathematics educators in Hungary. The authors introduce readers to the Pósa's technique of developing numerous methods as a means to foster student mathematical understanding. The authors illustrate the approach through exploration of multiple solution strategies for the* 1000 square task.

Keywords: *Problem solving, inquiry, discourse, teaching methods*

1 Introduction

In the Spring 2019 issue of OJSM, the authors presented the strategy "using a sequence of questions" that was shown at the *Budapest Semesters in Mathematics Education* (BSME) program which Bridget attended. The purpose of this article is to present a second strategy from that same experience.

1.1 Pósa and K-12 Mathematics

As an undergraduate student, Bridget had the incredible opportunity to participate in the BSME program. She met the famous mathematician Lajos Pósa (pronounced Posha) and studied his methods for teaching mathematics. Pósa is one of the most prominent mathematics educators in Hungary. His methods were originally developed for gifted students to discover concepts through individual exploration, but were soon used successfully with many different levels of students. One of the main aspects of his method is to have students experience how professional mathematicians might work on and think about mathematics in their careers. While this is a worthy goal for K-12 classrooms in the United States, how do teachers find or design tasks that engage students in activity akin to that of research mathematicians? Perhaps an answer lies in the recent emphasis on the use of "rich tasks" (also called cognitively demanding tasks) in K-12 classrooms in the United States.

1.2 Properties of Rich Tasks with the Standards for Mathematical Practice

What IS a rich task? What criteria should teachers use to decide whether to use a specific task for their class? In her book, *Modeling with Mathematics* (2015), Nancy Butler Wolf discusses six characteristics common to rich mathematical tasks that teachers should consider when deciding to use a task with students. These include:

- **Accessibility to all learners:** present tasks that provide opportunities for all learners to contribute with confidence.
- **Real-life tasks:** use tasks that are interesting and can help learners understand that mathematics concepts exist in the world beyond the classroom.

- **Multiple approaches and representation:** present tasks that allow students to approach mathematics from different perspectives, giving students more opportunities for success.
- **Collaboration and discussion:** use tasks that encourage students to gain new insights and perspectives by hearing from their peers; students gain confidence in problem solving when given the opportunity to explain their reasoning to others.
- **Engagement, curiosity, and creativity:** present tasks that are interesting and engaging to students. Students will persevere in problem solving when tasks inspire creativity and curiosity.
- **Opportunities for extension:** use tasks that include challenges and extensions for more advanced learners.

All of the characteristics in the preceding list also engage students in using many of the *Common Core Standards for Mathematical Practice* (CCSMP). For example, Wolf's characteristic of Collaboration and Discussion is highlighted in the CCSMP #3 where students are asked to "justify their conclusions, communicate them to others, and respond to the arguments of others," as well as CCSMP #6 where "students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning" (National Governors Association Center for Best Practices, 2010). Connecting the tasks that are chosen to the CCSMP's is a valuable and important aspect of effective teaching.

2 Pósa Strategy #2: Developing numerous methods

The second strategy from the BSME workshop that is presented in this article encourages students to develop as many ways as possible to clarify a given task as a prelude to generating multiple solutions. We believe that the problem presented below is an excellent example of a rich task as identified by the six characteristics listed above and connects nicely with many of the CCSMP's.

Consider, for instance, the following problem solving scenario.

1000 Squares Task: Can a square be divided into 1000 squares?

After a few moments,we ask students to pose follow-up questions that clarify the original task. Among these are questions such as: "Is 1000 a perfect square? Do all the squares have to be the same size? Do overlapping squares count? Can there be space left over, or do we have to use the entire square?" While many students may consider the original question to be too vague, an essential aspect of discovery-based learning is posing insightful and useful questions. Some questions, such as "Do overlapping squares count?" can be turned to the rest of the class to decide. This reinforces the idea that the teacher's role is no longer the source of all knowledge in a classroom. Students are given the power to discover and create mathematical questions and knowledge as individuals and as a team, rather than passively relying on an outside source (such as the internet or the classroom teacher).

3 Sample Solutions

After deciding that overlapping squares are not allowed and all the squares do not need to be the same size, the BSME class found several possible constructions over the course of a few days. We present several of them here as well as a few other possible constructions.

3.1 Perfect Square Construction

The first construction takes the square and divides it into 32^2 or 1024 squares of the same size. Notice that this is 24 squares too many. If you take a small 5×5 section, and "glue" the unit squares all together, you lose 25 small squares but gain one large one. Thus, there will be $1,024 - 25 + 1 = 1000$ squares, as shown in Figure 1.

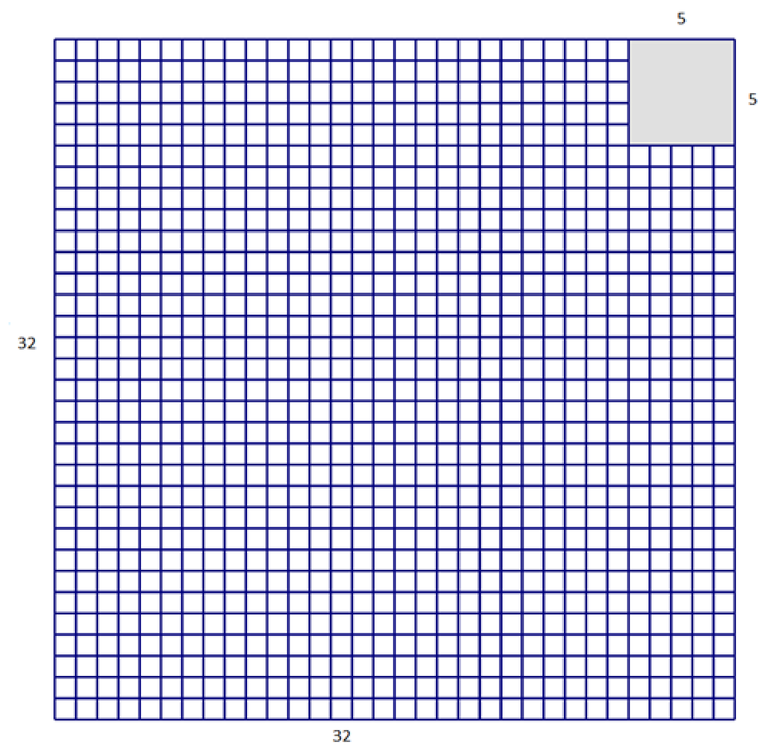

Fig. 1: *One construction of 1000 squares.*

3.2 Squares-in-Squares Construction

The second construction found by the BSME class was to create a sum of four square numbers which add up to 1000. One example is $20^2 + 20^2 + 10^2 + 10^2$. A pictorial representation of this scenario is shown in Figure 2.

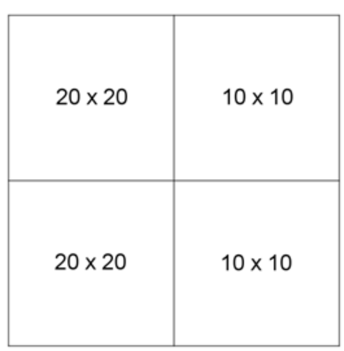

Fig. 2: *A solution with the square divided into quadrants.*

The square can be divided into quadrants, two quadrants are then divided into 20×20 grids, and the other two quadrants are divided into 10×10 grids. As students generate solutions to these, they recognized that there is not a unique sum of squares which create 1000. For example, $30^2 + 10^2$ also gives 1000. However this is impossible to draw since it requires at least two non-square sections. Thus, students discover that they will need four (or 9, or 16, or $25 \ldots$ etc.) initial square values in order break the square into smaller squares.

Furthermore, students can be directed to Lagrange's Four Square Theorem (also known as the Bachet Conjecture) which states that any natural number can be written as the sum of four square natural numbers. Unfortunately, although every number can be written as a sum of four square natural numbers, it is not the case that an actual diagram can be drawn for each number. For example, if we explore how a square can be divided into 17 squares, we find that $3^2 + 2^2 + 2^2 + 0^2$ will give us 17, yet this is impossible to draw.

3.3 Eva's Construction

When the authors tried this same problem with some actual K-12 students one of them completed the question similar to this last solution. Eva is a 6th grader who completed the question. The conversation that was held with Eva was as valuable as her solution. Eva was told the problem and she said "That's easy—each one is $\frac{1}{1000}$ the area of the large square." After being told that she had to design a way to fit them all together, she was not sure how to go about it. Thus, she was given a suggestion to start with a smaller number, like 8 or 10. She thought about it for a minute and said that she would put five squares on each side (shown in Figure 3).

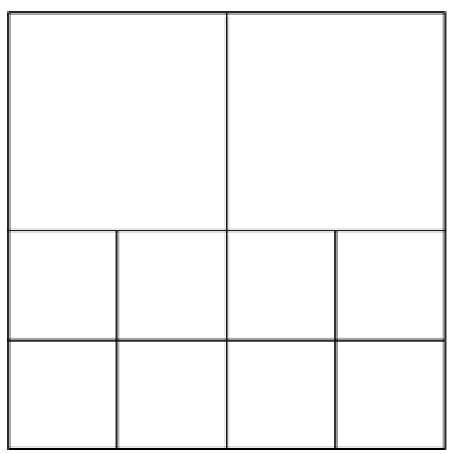

Fig. 3: *Eva's model for ten squares.*

Eva was then asked to try dividing one square into 100 squares. She thought about it for a few seconds, and said (excitedly) that she would divide each of her 10 squares the same way. Before too long, she then exclaimed that she could do that one more time and have 1000 smaller squares.

3.4 Bridget's Construction

A third construction which Bridget found requires starting by dividing the large square into fourths, as shown in Figure 4. Then one of those squares is divided into 23×23 squares, one is divided into 15×15 squares, and one is divided into 11×11 squares. This leaves a fourth square that is then divided into fourths again with one set of 6×6, one set of 9×9, and two sets of 2×2 squares.

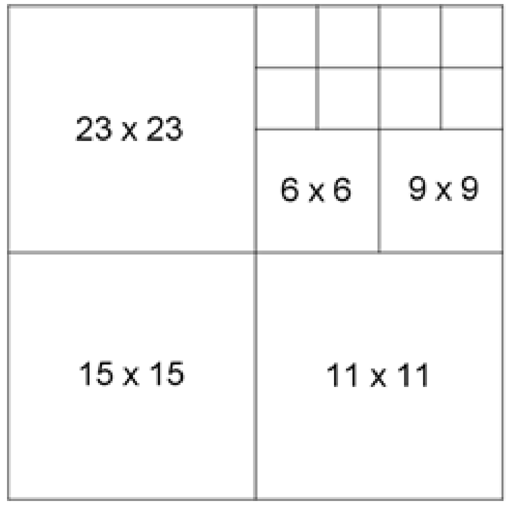

Fig. 4: *Bridget's model.*

3.5 Colloquium Construction

Another solution was found during a colloquium that Bridget presented to students at our institution. College students thought to break the problem down by dividing the original square into 4×4 squares (16 total), then making 14 of those into 8×8 squares, one of the original squares into 10×10 squares, and one of the original into 2×2 squares as shown in Figure 5.

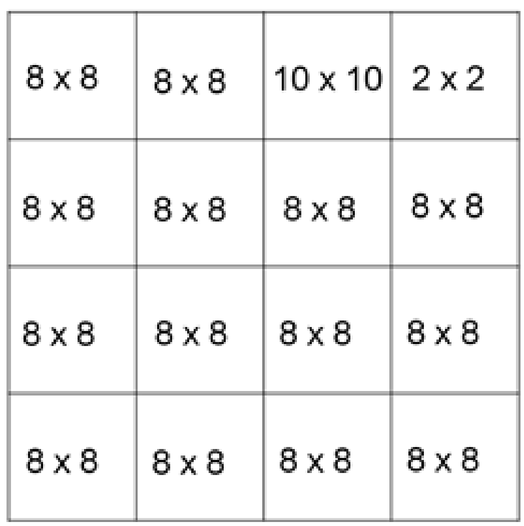

Fig. 5: *Colloquium Model.*

3.6 Fractal-like Construction

The authors also wanted to try to find a solution using a more structured approach with a fractal-like construction. In other words, we wanted to take the original square and divide it into fourths, then divide the upper right corner into fourths repeatedly as shown in Figure 6.

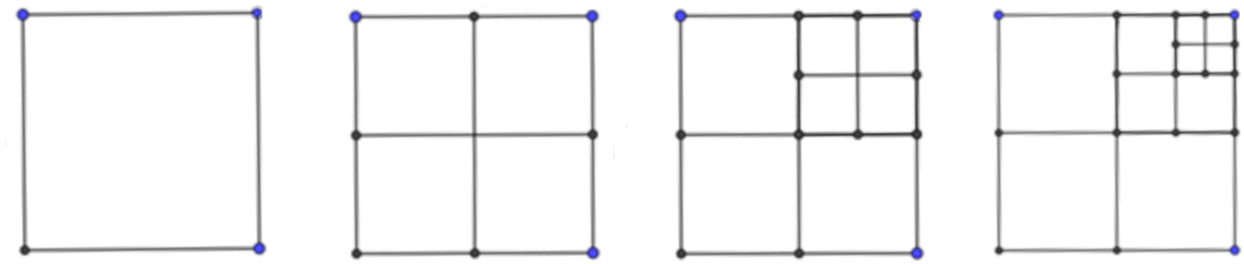

Fig. 6: *Fractal-like division of squares (first 4 iterations).*

Table 1 shows the number of squares formed for each iteration of the fractal.

Table 1: *Iterations of fractal squares*

Iteration (Step)	Number of Squares
0	1
1	4
2	7
3	10
4	13
5	16
...	...
n	$3n + 1$

This solution is systematic, and can be pursued as a way of thinking similar to the first scenario where students start to look for patterns and even make a table of values. Would this result in the square being divided into 1000 pieces? Yes. By solving $3n + 1 = 1000$, we determine that on iteration 333, there will be 1000 squares. What if each part is divided into nine pieces instead of four? Could we still get 1000 squares? (no) Through logical thought, students can determine which numbers can and can't be represented by a diagram of squares using this iteration method.

4 Additional Thoughts and Extensions

Each new construction reveals new numbers which can be represented in this way. Even more discussion can revolve around the numbers which can't be represented, and Lagrange's Four-Square Theorem (Bachet's Conjecture) can be shown as another mathematical connection. There are endless possibilities for this scenario and students should be challenged to find as many constructions as they can. What about trying to find a square that is divided into 500 squares? 200 squares? Obviously, the perfect squares are divided easily, like 100 squares are created by dividing into 10×10 smaller squares. It is the other numbers that are not perfect squares that are more difficult to divide.

5 Conclusion

Notice how this simple, eight word question can become an activity that meets all of Wolf's characteristics of rich tasks. The 1000 Squares Task is accessible to all students and allows multiple approaches, requiring students to work together and communicate their thoughts and solutions in an organized manner.

Unfortunately, teachers often do not have the time to spend on one problem that might require students several days to complete. Therefore, after introducing the task, this Pósa strategy might be best used for challenging students to find various solutions outside of class. Teachers will find that students will bring a new solution to them several weeks after the original discussion. The conversations that take place with this type of problem solving require students to be engaged and enhance their curiosity beyond what teachers could expect. These types of tasks have opportunities for extensions and give a real sense of how a mathematician might think about and work on a problem—exactly Pósa's objective.

References

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Note: The authors want to send a special THANK YOU to Eva for completing this task and to her parents for allowing us to use her solution.