Strings, Bears, & Boards: Making Fractions Meaning-Filled

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Abstract: *To successfully operate with fractions, decimals, and percents, pre-service teachers first need opportunities to make sense of the meaning of fractions. The Common Core State Standards and associated Progressions document provide guidelines for building fraction understanding. In this article, the authors present a series of fraction meaning activities developed through an iterative lesson study approach. The lessons use physical models—string, counting bears, and geoboards—to link pre-service elementary teachers' fraction reasoning to Common Core State Standards.*

Keywords: *fractions, manipulatives, lesson study, pre-service teachers, station teaching*

1 Introduction

1.1 The Importance of Fractions

Making sense of fraction meaning before fraction operations is foundational for students' development of rational number knowledge and fraction sense. The *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices, 2010) expects third graders to develop a rich understanding of fractions as numbers, explain their equivalencies, and compare fractions by reasoning about the size of pieces [Common Core State Standard 3.NF.A.1-3]. The need to understand fraction comparisons and operations continues well beyond high school through college and career settings—in algebra with proportional reasoning, transformations in geometry, and bivariate comparisons in statistics.

1.2 Fraction Meaning

Understanding fractions can be initially challenging, in part because one way of understanding a fraction involves coordinating two quantities using division and multiplication. Take the fraction $\frac{5}{3}$. One way to conceptualize this fraction is to first identify an unknown whole. This whole is then partitioned or divided into three equal parts, as suggested by the denominator. This part, which we refer to as $\frac{1}{3}$, is now iterated or copied five times, as indicated by the numerator (see Fig. 1). One general conceptualization is that $\frac{a}{b}$ signifies a copies of size $\frac{1}{b}$. This concept is reinforced in other countries, such as Japan, where the name of the denominator is pronounced first, followed by the number of copies under consideration (i.e., the numerator). Five-thirds is translated as "thirds taken five times."

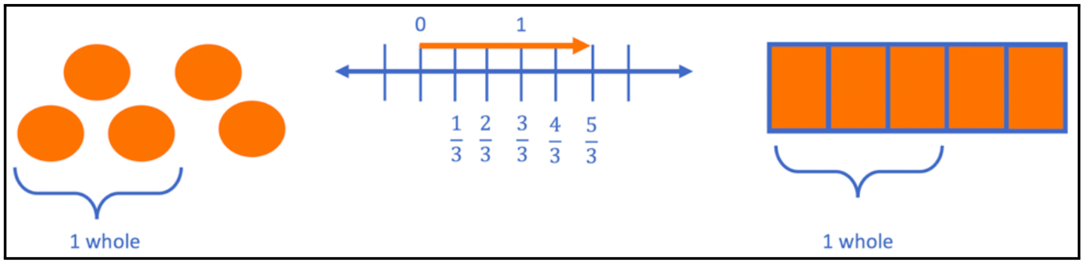

Fig. 1: *The fraction* $\frac{5}{3}$ *viewed as five copies of the unit fraction* $\frac{1}{3}$ *using discrete, linear, and area models.*

1.3 Developing Activities through Lesson Study

To address concerns around pre-service teachers' preparation to teach fractions (Luo, Lo, & Leu, 2011), we embarked on a multi-year lesson study to cyclically design, implement, debrief, and improve an introduction to fractions lesson. Lesson study is a collaborative, cyclic form of lesson improvement that involves a small group of instructors studying, planning, teaching, observing, and debriefing a research lesson (Lewis, Perry, & Hurd, 2009). While specific goals may vary, the goal of lesson study is to better understand student learning about a particular topic. Our lesson study team united mathematics teacher educators across two departments at our university: mathematics and elementary / bilingual education. The first cycle of lesson study occurred in Spring 2016, with nine additional iterations occurring to the present.

As we introduced fractions to pre-service teachers (PSTs), our lesson study team wondered which activities would promote rich mathematical reasoning and build productively on candidates' knowledge of whole number operations (Bezuk & Cramer, 1989). Through lesson study, we had the opportunity to design, test, improve, and reflect on a series of activities that built strong foundational understanding of fractions. Using several different models and contexts to represent mathematics, explicitly connecting fraction symbols to both physical and written models, and developing questions to push student thinking about each activity, we hoped to support PSTs in developing a conceptual understanding of the meaning of fraction.

In this article, we present three carefully designed activities that have been used with both PSTs and practicing elementary teachers to unpack the meaning of fraction standards. Implemented as three concurrent stations or centers, these activities support the development of important fraction concepts that lay the foundation for comparing fractions, understanding the relationship between mixed fractions and improper fractions, and operating with fractions and decimals. Groups started at different stations and spent approximately 15 minutes at each. All students experienced each station by the end of class. Note that the order of stations did not matter since each activity involved using a different manipulative to reach a similar learning goal. While two co-teachers taught this lesson, readers may alternatively choose to implement each activity as an individual lesson to fit the needs of their own classroom.

1.4 Learning Goals

Before planing the fraction activities, we used the Fraction Progressions document linking Grades 3 through 5 fraction standards to guide lesson design (Common Core Writing Team, 2013). Our lesson study began by carefully constructing the following learning goals:

1. **Meaning of fractions**
    a) **Same sized pieces:** Understand the fraction $\frac{1}{b}$ as the quantity formed by one part when

the whole is partitioned into b equal parts [Common Core Standard 3.NF.A.1]. Recognize that equal shares of identical wholes need not have the same shape [Common Core State Standard 2.G.A.3].

b) **Iterating unit fractions:** Understand the fraction $\frac{a}{b}$ as the quantity formed by a parts of size $\frac{1}{b}$. In other words, a indicates the number of copies of the unit fraction $\frac{1}{b}$ [Common Core State Standard 3.NF.A.1].

2. **Different representations of fractions:** Understand three diagram models for representing rational numbers: linear, area, and discrete. In other words, the whole can be a shape (circle, rectangle, etc.), a line segment, a collection of objects, or any one finite entity susceptible to subdivision and measurement [Common Core State Standard 3.NF.A.2].

3. **Specifying the whole:** Though not present in the Common Core State Standards, we also determined a third learning goal that we called "specifying the whole"—understanding that the value of a fraction depends on the whole. In other words, the size of the pieces depends on the referent whole (Philipp & Hawthorne, 2015).

Together these learning goals guided our lesson design, implementation, and assessment. We now present three activities that can be used to support PSTs' understanding of the meaning of fractions. We additionally report on how each activity was improved through the lesson study process.

2 Activity 1: Number Line Station

PSTs were stationed in a group near a four-foot long number line with a set of dangling fraction tents (see Fig. 2). Groups were given a set of task cards, each indicating two fractions to place on the number line, and a third fraction to place as accurately as possible based on the position of the first two fractions. For example, one prompt said, "If this is where 0 and 1 are, show $\frac{2}{3}$." To place $\frac{2}{3}$, PSTs could place $\frac{1}{3}$ and iterate two copies of it. We varied the types of questions posed, often asking PSTs to iterate to a number outside of an interval of two given numbers (e.g. "If this is where 0 and $\frac{3}{7}$ are, show $\frac{5}{7}$), or partition down to a number within an interval of two given numbers (e.g. "If this is where 0 and $\frac{5}{3}$ are, show 1"). The instructor of this activity asked groups to justify their placements using the meaning of fractions, specify their whole, and show how they made use of iteration.

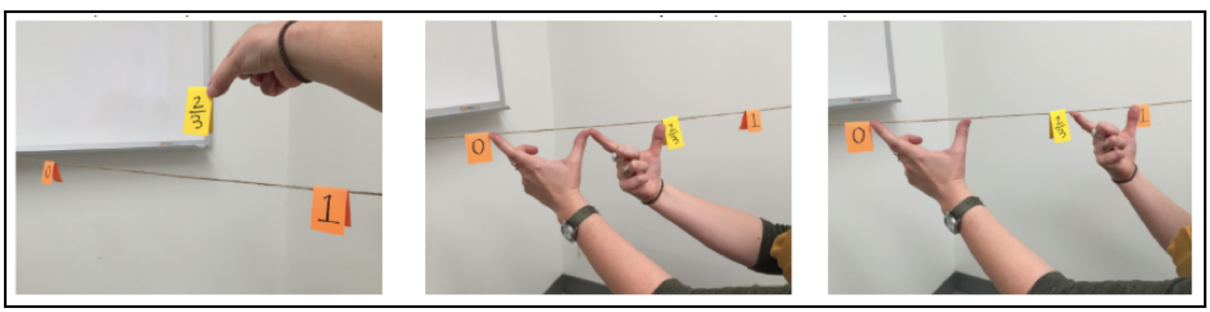

Fig. 2: *Iterating one-third to accurately place two-thirds.*

2.1 Improving the Number Line Activity

This activity was improved through the lesson study process in a number of ways. First, the instructor observed a number of student groups rotating through the activity and became more adept at anticipating student thinking, asking improved questions. Below is a sample interaction between an instructor and a PST at the number line activity:

INSTRUCTOR : I see you placed $\frac{1}{2}$ in the middle of 0 and 2. Could you say more about that?
PST : We placed $\frac{1}{2}$ in the middle because it's half way.
INSTRUCTOR : Where would you place 1 on this number line? (*PST attempts to place 1 where the $\frac{1}{2}$ is already located. PST appears to be thinking.*)
INSTRUCTOR : In this situation, when we say $\frac{1}{2}$ do we mean $\frac{1}{2}$ of 2?
PST : I think it means $\frac{1}{2}$ like where $\frac{1}{2}$ is located.
INSTRUCTOR : So where would that go?
PST : Here? (*Places $\frac{1}{2}$ between 0 and 1.*)

Clarifying that the unit interval from 0 to 1 is the whole supported student reasoning with the number line. Next, we noticed that some PSTs began by ordering the entire set of fraction tents rather than those described on the task card. We thought this was important to address due to our learning goals of partitioning and iterating unit fractions (rather than comparing fractions). Consequently, we clarified directions by asking PSTs to clear their number line after each task.

3 Activity 2: Geoboards Station

In this activity, PSTs worked with individual 5 peg by 5 peg geoboards (i.e., boards with side length equal to four units). After inviting PSTs to play with the physical manipulative and asking how geoboards could be used to think about fractions, the instructor asked PSTs to divide the geoboard into four parts. Some split their board into four equal parts while others split them into four non-equal parts (see Fig. 3). The instructor facilitated conversations about PSTs' representations of four parts. Next, the instructor asked PSTs to show fourths. Some updated their boards while others kept their original designs. We found it useful to have examples of congruent fourths, non-congruent fourths, and non-fourths ready for PSTs to debate. In each situation, PSTs developed an argument that proved whether each piece represents a fourth of the board.

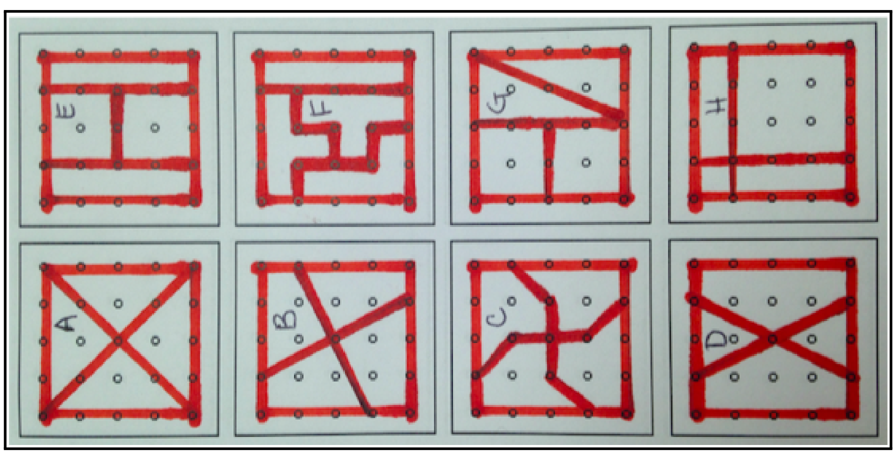

Fig. 3: *Geoboard fourths (A, B, C, E, F, G), non-congruent fourths (E, F, G), and non-fourths (D, H).*

We found that an interesting conversation frequently occurred:

PST : Do they have to be *equal* fourths?
INSTRUCTOR : Can you say more about what you mean?
PST : Do the fourths all have to *look the same*?

In this conversation, the phrase "equal fourths" was used to communicate congruent fourths. Discussing the difference between congruent fourths versus non-congruent fourths was useful at this point to determine how it related to the area of each piece. We found this distinction to be important: PSTs realized that while fourths of the same whole do not need to be the same shape, they need to have the same size (e.g., the same area in the area model). We determined that while “equal fourths” was a redundant phrase, “congruent fourths” gave more specificity about the situation, pushing PSTs to convince one another that each fourth was the same size. Another interesting question we asked them was, “Can you partition your geoboard into fourths such that none of the fourths are congruent to each other?” PSTs (and instructors!) enjoyed working on this challenge question.

3.1 Improving the Geoboard Activity

Our lesson study team focused on the question, “Is ‘equal fourths’ the same thing as ‘fourths’?” This question allowed conversation about same-*shaped* pieces versus same-*sized* pieces. This also helped us distinguish between “four pieces” and “fourths.” We believe it is important to identify “equal fourths” as redundant—to say “fourths” is, by definition, synonymous with “four equal pieces.” As a result, we asked PSTs, “What is the difference between the size of a piece versus its shape? What does congruent mean in this context?” To encourage discussion, we created pre-made boards depicting various constructions of “fourths” to push PSTs’ thinking (see Fig. 3). For instance, if they said, “So long as each piece has the same number of pegs in the interior, then the size of each fourth is the same,” we would provide a geoboard that showed fourths but without the same number of interior pegs in each piece. Interestingly, this idea is related to Pick’s Theorem, which gives the area of polygons on a geoboard given the number of pegs within the interior and boundary of the polygon (Russell, 2005).

4 Activity 3: Counting Bears Station

In this independent activity, PSTs solved tasks with counting bears as physical manipulatives. Tasks asked PSTs to find the number of bears in a forest given some initial fractional amount of bears (see Fig. 4).

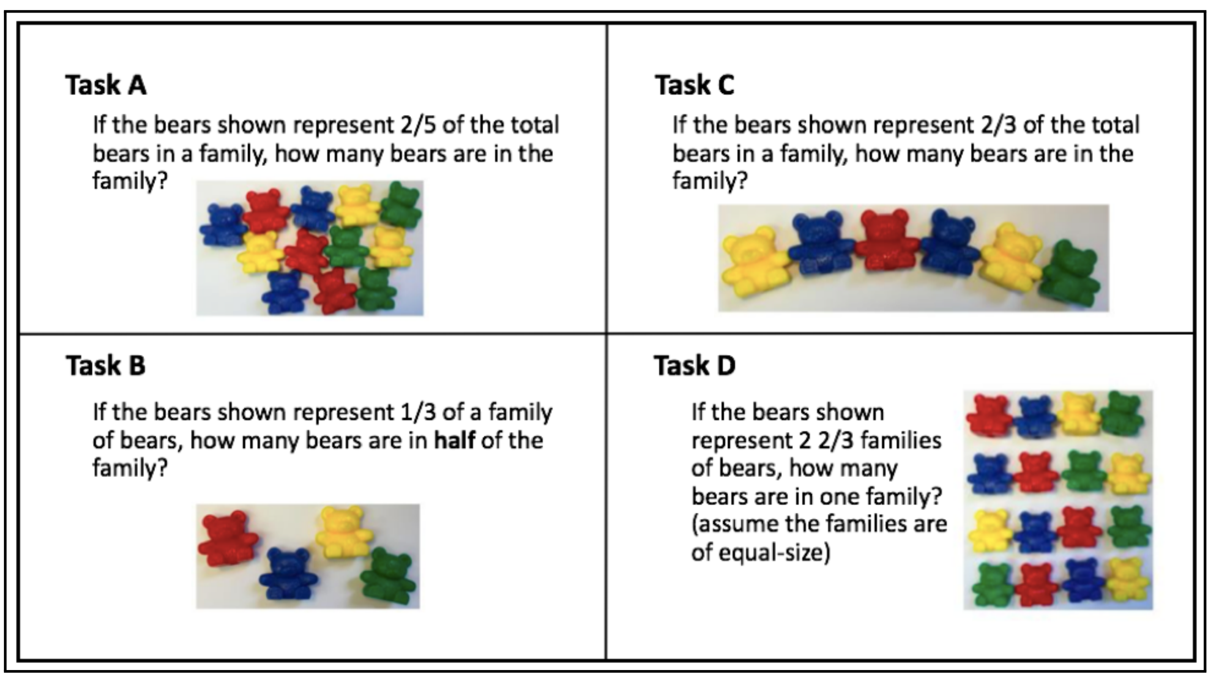

Fig. 4: *Task cards for counting bears activity.*

This activity brought together the ideas of partitioning and iterating. For instance, to find how many bears are in one family if 16 bears shown on the card represent $2\frac{2}{3}$ families of bears, a PST might first find an equivalent fraction to $2\frac{2}{3}$, namely $\frac{8}{3}$, and notice that the given 16 bears need to form eight equal groups of something. Then, a PST might notice that with two bears in each group, eight groups of two bears form the 16 shown bears. This implies that three groups of two bears create one family. This task can also be solved using ideas of proportional reasoning: for every third of a family, we have two bears.

4.1 Improving the Counting Bears Activity

We noticed that the language of our original prompt was inconsistent. Sometimes we referred to a group of bears as a *family*, sometimes as a *forest*, and then other times as a *group*. Thus, in our revision we decided to use the unit of *family*. The second main issue centered on PSTs' use of manipulatives. The first time we taught this lesson, few used the manipulatives to explain their reasoning. Most PSTs wrote proportions on the paper to solve the task symbolically. Consequently, we did not provide paper for future iterations of this lesson. In doing so, most PSTs touched the manipulatives, which helped reach our learning goal of understanding how to model fractions with different representations.

5 Discussion

Lesson study provided a structure for rich discussion regarding the design and implementation of fraction activities aligned to *Common Core State Standards*—including viewing $\frac{a}{b}$ as a copies of $\frac{1}{b}$, iterating unit fractions to build fractions, understanding three different representations for modeling fractions (linear, area, and discrete), and the importance of specifying the whole when discussing fractions. These foundational concepts extend to discussions on percents, decimals, ratios, and statistics. Using carefully designed activities, we provided PSTs complementary learning experiences that targeted important concepts around the meaning of fractions.

We share this example of mathematics teacher educators engaging in lesson study on the meaning of fractions with their pre-service elementary teachers for a number of reasons: (a) to illuminate several activities that can be used to learn the meaning of fractions as portrayed by *Common Core State Standards* and (b) to demonstrate the process of and reasons for engaging in lesson study. We hope this reflective process provides inspiration for instructors to inquire into their own practice to improve student learning.

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