Standardized Test Questions Involving Fractional Diagrams: What do they Assess and How Can Teachers Effectively Use Them in Class?

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Abstract: *Many standardized tests have begun to broaden their assessment beyond the use of symbolic representations and now include more concrete diagrams. This paper reports on the conceptual resources 5th grade students leveraged when attempting to answer such questions in the domain of fraction operations. Results demonstrate how such questions failed to assess students' understanding of the diagrams or of fractions, as the vast majority of participants used symbolic procedures or superficial qualities of the diagrams to navigate their answer choice. However, student responses, when pressed to explain their thinking about the diagrams, revealed rich thinking, illustrating how such questions can serve as productive formative assessment tasks.*

Keywords: *standardized testing, representations, early grades, conceptual understanding*

Introduction

Diagrams play an important role in the learning and understanding of mathematics (NCTM, 2000). They support flexible problem solving, helping students construct meaning of problems and explore alternative solution methods (Stylianou, 2011). The *Common Core State Standards for Mathematics* (2010) highlight the importance of diagrams, outlining their use in multiple content areas. In the domain of fractions, the standards explicitly denote that the understanding of each operation involves the use of visual models to capture the underlying relationships. An emphasis on non-symbolic representations is not surprising as scholars have argued that students cannot develop a conceptual understanding of the symbolic manipulation of fractions without also possessing the ability to solve such problems using more concrete, visual representations (Battista, 2012; Huinker, 2015).

In an attempt to align themselves with these new standards, authors of standardized tests have begun to broaden such assessments beyond traditional symbolic representations and include the use of more diagrams. For example, in the last two years on our district's tests, 40% of the problems assessing students' understanding of fractional operations featured a diagram. Given this proliferation of diagrams on such tests, we were curious to know what the standardized questions involving diagrams assessed. Moreover, if students used the diagrams while solving our district's standardized test problems, how were they used and how could their responses interpreting the diagrams inform us on how to use such questions in class?

Methods

To explore these questions, we interviewed 20 fifth grade students to see what resources they drew on to answer a collection of standardized test questions involving diagrams. In an attempt to ensure a broad range of thinking, participants were drawn from a highly diverse school (53% Free and Reduced Lunch; 22% Gifted and Talented). In terms of race, there were eight white, six black, one Hispanic, one Native American, and three international students. In addition, students came equally distributed from three different classes and were intentionally chosen to represent a broad range of mathematical ability, based on the teachers' perception of aptitude. During the interview, each student was asked to explain how they initially solved the problems. They were then asked a series of questions targeting their understanding of the diagram. From these interviews, two findings emerged.

Findings

First, we found that almost no student used the diagrams to solve the problems. Instead, they showed a preference for using more traditional symbolic representations when possible. Even when the question required students to engage with the diagram, they relied on superficial qualities to guide their thinking, rather than use the diagram to solve the problem. Second, when we followed up with students and pressed them about their understanding of the diagrams, their responses revealed details of their understanding around key mathematical ideas. Engaging with the diagrams encouraged students to grapple with core fraction concepts. To illustrate the wide divide between the knowledge students used to answer the questions and the understanding that emerged when asked to make sense of the diagrams, we highlight student responses on three fraction questions. In addition, we note ways such questions might be adapted to be used more productively as a tool for teachers to elicit and support students' understanding.

Illustration 1: Multiple Choice Area Model

The first question presents an area diagram illustrating multiplication of mixed fractions and asks students to select the number sentence represented in the diagram (see Figure 1). Accompanying the diagram is a series of numbers that indicate how wide and how long the diagram is. Each answer choice includes the same number sentence expression with different answers.

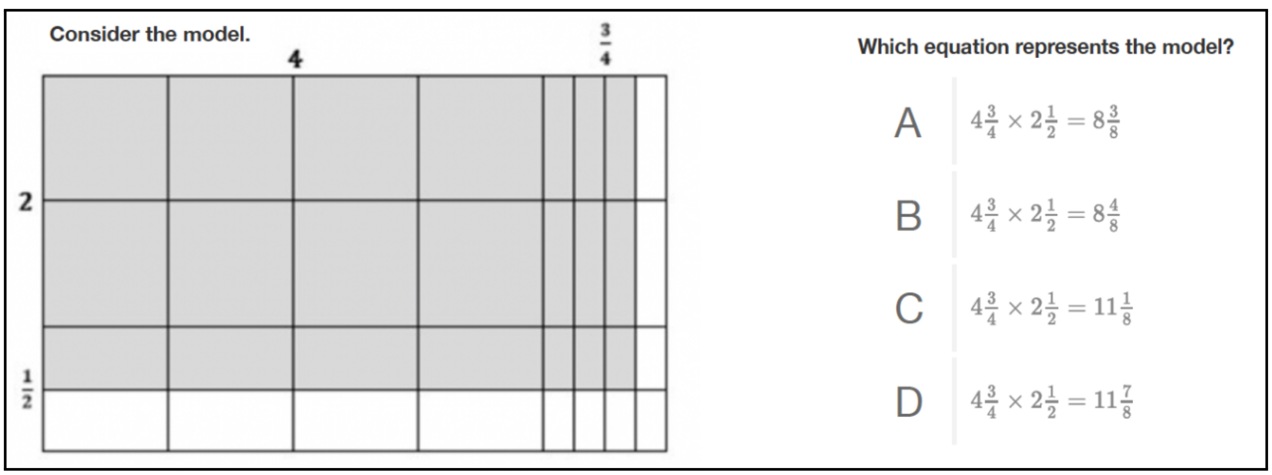

Fig. 1: *Question 1—Mixed number multiplication-distributive property.*

Reviewing the student interviews, we found that of the 20 students, only 5 attempted to use the diagram. Of the others, 2 simply guessed and 13 used a symbolic procedure without thinking of its meaning in the diagram. Of these 13 students, 4 correctly converted the numbers into improper fractions to create $\frac{19}{4} \times \frac{5}{2}$, and multiplied the numerators and denominators. The other 9 multiplied the whole numbers and fractional parts separately to create 4×2 and $\frac{3}{4} \times \frac{1}{2}$, resulting in a wrong answer. While this problem contains a diagram, it did not assess students' understanding of this diagram since the majority of students operated with symbols. It seemed that the structure of the problem encouraged students to use a symbolic procedure over a more meaningful approach involving the diagram, in effect lowering the cognitive demand of the task. With all of the multiple-choice options consisting of the same number sentence with only the answer differing, students engaged in the problem as a procedure without connections (Smith & Stein, 1998).

While the structure of this question failed to measure any fractional understanding beyond the ability to carry out procedures, students did begin to grapple with fundamental concepts associated with fraction multiplication when pressed to interpret the diagram. As they worked through different decomposed parts of the multiplication, their responses revealed a differentiation in their understanding of such concepts as the repeated addition associated with the multiplication of a whole number and a fraction as well as the operator view of fraction multiplication. As such, while this question failed as a summative assessment to evaluate students' understanding of fraction multiplication using a diagram, the richness of thinking that emerged as students described their understanding of the diagram highlights that this question can serve as an effective tool for formative assessment with only a few adjustments. For example, a teacher could introduce the diagram as a possible student solution to $4\frac{3}{4} \times 2\frac{1}{2}$ and ask the class how the student might have been thinking. Without multiple choice responses, this could lead to a discussion where students grapple with how the diagram illustrates the distributive property, connecting their understanding of mixed fraction multiplication with base 10 whole number operations. Such an activity would elevate the cognitive demand and provide invaluable information to the teacher.

Illustration 2: The Bakery's Cake

The second question presents a word problem with four choices of area models and asks students to select the diagram that represents the correct solution (see Figure 2).

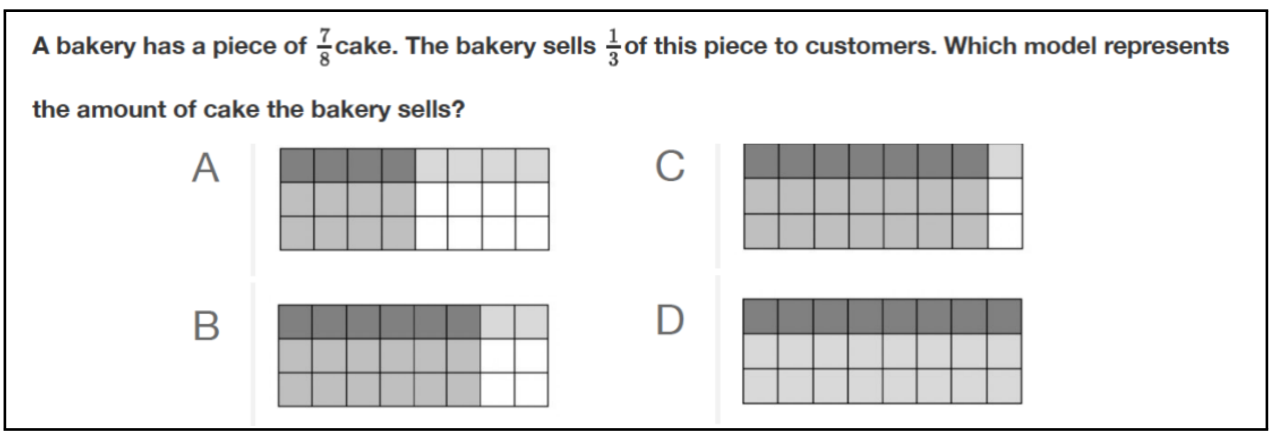

Fig. 2: *Question 2—Fraction multiplication-operator view of fraction multiplication.*

In contrast to Question 1, this format inherently requires students to engage with the diagrams since symbolic representations of solutions are not available as resources. However, while 16 of 20 students identified C as the correct diagram, we found that not one student actually used the diagram to solve the associated word problem when deciding on their answer choice. Instead, they simply looked for possible representations of $\frac{1}{3}$ and $\frac{7}{8}$ within the diagram without reflecting on their meaning or considering their product. For example, Annie (all names are pseudonyms) chose C because she noted that 7 of 8 blocks in the top row were shaded. When asked if she used the diagram to find the amount of cake sold, she said no and explained that "I knew right away when the 7 was colored that was the right one. But after that I kind of trusted it." Similarly, Thomas noted that in C, in addition to the top row being shaded, 7 of the 8 tiles in the bottom two rows were shaded. When asked to explain what the shaded part in top row represented, he offered a highly decontextualized answer, simply reiterating that it was the "1 over the 3." Finally, when pressed to use the diagram to solve the multiplication problem, he responded, "It would be hard. I would have to use the . . . numbers," and then illustrated this using a symbolic procedure.

Ultimately, five students were able to use the diagram to calculate the answer to the word problem when asked to do so. However, three of these students did so by treating the entire rectangle as the whole for both $\frac{7}{8}$ and $\frac{1}{3}$ and using their intersection to find the numerical solution. Although the intersection does provide the correct answer, such a method does not make meaningful sense in terms of the context (Philipp & Hawthorne 2015; Webel, et al., 2016). The rectangle is the correct whole for the $\frac{7}{8}$, while the appropriate whole for the $\frac{1}{3}$ is the $\frac{7}{8}$ remaining piece, not the entire cake. Such a distinction is at the core of understanding fractional multiplication, but one that was not assessed by the problem, nor understood by the students. When asked while interpreting the diagram if it mattered that it was "$\frac{7}{8}$ of cake" and "$\frac{1}{3}$ of the piece," the students all said no. Even when presented an alternative diagram which incorporated this distinction (see Figure 3), they characterized the diagram as incorrect, asserting that it did not accurately show $\frac{1}{3}$. Only one student saw the alternative diagram as correct. However, she actually skipped the problem, believing all of the answer choices were incorrect. Such responses highlight that these students, just like the others, were not attending to the referent unit or how it changes throughout the problem.

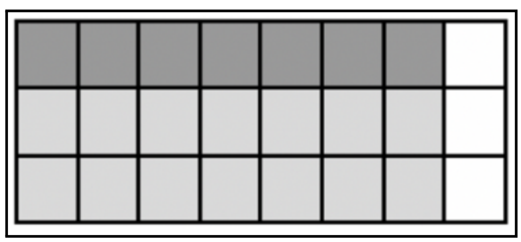

Fig. 3: *Alternative diagram for Question 2.*

The Bakery Cake question, as constructed on the standardized test, once again raises questions about the effectiveness of using diagrams within the multiple-choice format as a summative assessment tool. Even with the diagrams included as the answer choices, this question failed to assess students' understanding of fraction multiplication as the majority of students were able to identify the correct diagram by using superficial qualities to guide their answer choice. Such a disconnected view from the context seems almost encouraged by the design of the problem as none of the diagrams accurately depicts the quantities and their relationships. Option C was considered the correct answer choice, but in our eyes, it incorrectly refers to the whole of $\frac{1}{3}$ as the whole cake. In fact, the diagram in C embodies the qualities associated with this intersection method which has become a prevalent way to proceduralize the use of diagrams (Webel et al., 2016). Unfortunately, in problems such as the Bakery Cake context, the intersection method fails to support an accurate and robust understanding of fraction multiplication.

Nonetheless, the interviews again highlighted that when accompanied with follow-up questions, this problem could serve as a productive formative assessment tool to reveal how students are conceptualizing the referent unit of these fractions. Rather than stopping after students provide their choice of diagram, teachers can follow-up and probe students about their interpretation of the diagram. Specifically, they can ask where the $\frac{7}{8}$ of the cake and $\frac{1}{3}$ of the piece are in the diagram and if it matters that the two fractions are referring to different parts of the cake. Having students share these different interpretations can provide the basis for a rich discussion about the role of the referent unit in fraction multiplication as well as offer teachers useful insight into students' understanding.

Illustration 3: Equal Sharing View of Fractions

In addition to multiplication, we also looked at equal sharing problems that assessed students' understanding of the quotient view of fractions. Figure 4 provides an example in which students were given a word problem with four different diagrams as answer choices and asked to pick which diagram gives the correct solution to the word problem. We found this question particularly interesting because depending on students' interpretation of the diagram, there are two possible correct answers, A or C. This allowed us to see what characteristics of the diagram were salient for the students as they made their selection.

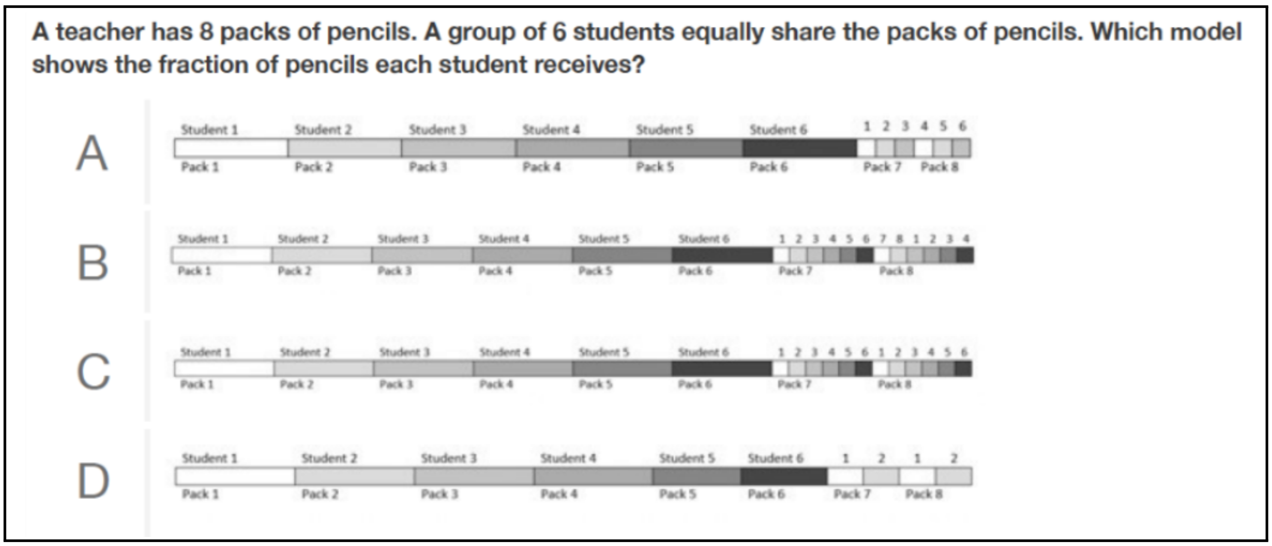

Fig. 4: *Question 3—Equal Sharing Context (Quotient View of Fractions).*

Similar to the previous illustration, when explaining their answer choices, no student attempted to find the fractional answer $1\frac{2}{6}$ (or $1\frac{1}{3}$) of pencil packs during the initial problem-solving process. Instead, they looked for details in the diagram they believed captured the distribution process, often relying on superficial features to guide their thinking. For example, Jalen and Daniel used the colorings of the diagram and the size of packs to determine their answer. Jalen argued that answer A would not be correct, explaining that not all students would get the same number of pencils because "the colors do not line up." Similarly, Daniel reasoned that answer choice A would not be correct because the last two packs in A were visibly smaller and as such would only have "3 crayons in each pack."

As these examples highlight, students made their choice based on how effective they perceived the diagrams demonstrated that each person received the same amount of packs and pencils. Using such thinking, all but 3 students were able to select a correct choice of A or C. However, while this reasoning demonstrates an understanding of how the drawing connects to the equal sharing context, it does not reveal students' understanding of fractions as a quotient, since no student actually grappled with the fractional amount represented.

Similar to the previous problems, when pressed to explain their thinking, students' interpretation of these diagrams provided rich insight into their understanding of fractions as a quotient. For example, Katia was able to not only explain how the diagrams in A and C illustrate how students receive $1\frac{1}{3}$ and $1\frac{2}{6}$ pack of pencils, respectively, but also clearly articulated how the distribution processes and associated symbolic forms differ but result in equivalent amounts. (Of note, Katia would not have received credit on the test as she initially chose A, thinking the diagram with $1\frac{1}{3}$ demonstrated a more efficient method.)

Other explanations revealed areas where students still had areas for growth. Most notably, 40% of the students conceptualized the answer in terms of whole numbers, suggesting a lack of understanding of fractions as a quotient. To do so, they assigned a certain number of pencils to each pack. Johnny used this approach to explain how A and C were both correct, but resulted in students receiving different amounts. He believed that because the packs were different sized and were subdivided into a different number of pieces, that diagram A indicated that each student would receive 1 pack and 1 pencil, while students in diagram C would each receive 1 pack and 2 pencils. Other students struggled conceptualizing the whole as they treated packs 7 and 8 as a collective unit. For example, Daniel explained that diagram C showed that each student would receive 2 and $\frac{2}{12}$ of a pack since there were a total of 12 pieces.

While many students voiced a range of conceptions about the amounts represented, these interpretations initially went undetected. Such inconsistencies further highlight how such problems involving fractional diagrams appear to be ineffective as a summative assessment. However, students' rich interpretations of the diagrams indicate that such a problem can still be used productively as formative assessment. To do so, teachers simply need to follow-up students' initial choice and probe how they interpret the diagram, asking specifically how many packs each student receives according to their interpretation. Moreover, by using students' responses as the basis of mathematical discussion, the current ambiguity in diagrams becomes an asset, as the class grapples with how both A and C represent different meaningful solutions.

Conclusion

Though test makers strive to broaden standardized testing by incorporating more concrete representations, this study illustrates how such questions, in the multiple-choice format, might not successfully assess the intended understanding. While we hope these findings can help those who design and use such high-stake assessments, we are hopeful that teachers can draw from these results to better decide how to use such questions in their classroom.

An important takeaway for teachers is that when used as a summative assessment with attention only focused on a final score, such questions might reveal little about students' understanding. The majority of participants who answered these questions correctly did so without grappling with the targeted understanding of diagrams or fractions. Moreover, some students were even penalized as their productive insight led them to choose answers that were deemed incorrect. However, these illustrations also show that such problems, when used as formative assessment and accompanied with probing questions, can be used to effectively explore student thinking about fraction reasoning.

As the interview responses revealed, the embedded diagrams can serve as rich tasks, uncovering a wide range of student reasoning. As such, teachers do not need to search out and create new problems. They simply need to ask students how they are interpreting the diagrams and using them to solve the problem. We believe such questions can productively modify these problems, providing a rich foundation for meaningful discussions around fraction operations.