# Base 3 Numeration System Requiring a Negative Digit

**Steven Meiring,** *Ohio Department of Education (retired)*

***Abstract:*** *An investigation of the minimum number of weights for weighing objects on a two-pan balance leads to two interesting discoveries. The first is that the weights must all be powers of three. The second is that a model to describe the process requires a base three numeration system with a negative digit.*

***Keywords:*** *problem solving, critical thinking*

## Introduction

A compelling aspect of mathematics is its power to raise intriguing questions from the real world. Sometimes these questions are not the original one asked. But rather ones that arise in the mind of the investigator, following the original solution, that say, I wonder why this must be?

One such arose in connection with the following cute balance problem, last posed in the article "Stella and the Stunners: The Mathematical Legacy of Rudd Crawford," published in the Fall 2020 issue of this Journal. Before turning to the question which forms the basis for this article, let's start with the original balance problem and how it is resolved – since it supplies a clue to answering the more challenging puzzle to follow.

## Weights and Balance

**Martha the Grocer**

Martha has a 2-pan scale and a set of 4 shiny brass weights. She claims that she can weigh out any whole number of ounces up to and including 40 using just these weights. And she's right! What is the weight of each of her 4 weights? (Note that you can put weights on either pan of the scale.

In determining an object's weight, we readily perceive that weights added to the pan of the object being weighed must be subtracted from the weights on the other pan. Through trial and error (and perhaps a good deal of experimenting), we discover that weights which are successive powers of 3 – namely 1, 3, 9, 27 – will enable us to weigh any object from 1 - 40 ounces. The full solution to this problem is demonstrated in Table 1.

**Table 1:** *Data for the Balance Problem*

| Range by Powers | Weight Combinations | Weights Available | Amounts that can be Weighed |
|---|---|---|---|
| | 0 | none | 0 |
| $3^0$ | $1 = 3^0$ | 1 | 1 |
| $3^1 - 1$<br><br>$3^1 + 1$ | $2 = 3^1 - 3^0$<br>$3 = 3^1$<br>$4 = 3^1 + 3^0$ | $1, 3$ | 1 to 4 |
| $3^2 - 4$<br><br><br><br><br><br><br><br>$3^2 + 4$ | $5 = 3^2 - 3^1 - 3^0$<br>$6 = 3^2 - 3^1$<br>$7 = 3^2 - 3^1 + 3^0$<br>$8 = 3^2 - 3^0$<br>$9 = 3^2$<br>$10 = 3^2 + 3^0$<br>$11 = 3^2 + 3^1 - 3^0$<br>$12 = 3^2 + 3^1$<br>$13 = 3^2 + 3^1 + 3^0$ | $1, 3, 9$ | 1 to 13 |
| $3^3 - 13$<br><br><br><br>$3^3 + 13$ | $14 = 3^3 - 3^2 - 3^1 - 3^0$<br>$\ldots$<br>$27 = 3^3$<br>$\ldots$<br>$40 = 3^3 + 3^2 + 3^1 + 3^0$ | $1, 3, 9, 27$ | 1 to 40 |
| $3^4 - 40$<br><br><br><br>$3^4 + 40$ | $41 = 3^4 - 3^3 - 3^2 - 3^1 - 3^0$<br>$\ldots$<br>$81 = 3^4$<br>$\ldots$<br>$121 = 3^4 + 3^3 + 3^2 + 3^1 + 3$ | $1, 3, 9, 27, 81$ | 1 to 121 |

## Questions after the Solution

> Solving one problem always uncovers new problems that we could not have discovered before. Answering one question inevitably leads us to new and better questions.
>
> *The Quotable Teacher*

The original balance problem is accessible to virtually any secondary level student and is a nice little challenge for thinking differently about how weights can be used on a balance. This author has tried to instill in students the notion that you are not finished with a problem until you have written a ***Take Away.*** Namely, what have you learned as a result of solving this problem that you did not know before (about mathematics, about thinking, about yourself)? High on the list of possible Take Aways are related questions that occur to the solver, in the spirit of Pólya's *Looking back heuristics.*

For the more experienced problem solver, who looks back at the solution, numerous, more formidable questions emerge: Why powers of 3? If we add another weight, what range could be weighed? How do we know? In general, if the weights are

$$3^0, 3^1, 3^2 \ldots, 3^n$$

what range could we weigh? We leave to the reader these discoveries from Table 1 with one exception. The first question intrigues us. Why powers of 3 instead of some other base? And how, exactly, does this work for the balance? (If you are so inclined, stop reading here and go off to ponder this question until you are satisfied. It took us a few days.)

Certainly, with a base 3 numeration system, we would be able to weigh any amounts that we could in base 10. That realization eventually leads us to consider how powers of 3 could be used as place values in a representation such as a hand counter (as illustrated in Figure 1). For base 3, we need the numerals {0, 1, 2} for counting in each place-value position. A condition which we seem not to have, since we have only one weight for each power of 3.


**Fig. 1:** *Hand counter.*

Eventually, after our brain has the opportunity to incubate the question, we come back round to considering how a weight may be used. We then recognize that it can be used in three ways. On the right side, for which its weight contributes a factor of (1). Not used at all, representing a factor of (0). Or used on the left side with the object being weighed —in which case its weight must be subtracted from those on the right pan, thereby representing a factor of (-1).

So, we do have three states for each weight {-1, 0, 1} just not what we were hoping for {0, 1, 2}. Then one of those beautiful moments in mathematics arrives with the aha! realization that $\{0,1,-1\} \equiv \{0, 1, 2\}$ mod 3. Then we suddenly understand why a base of 3; and, also how the counter must work.

The numerals in order for each dial for our counter will be {-1, 0, 1} in that order.

| $3^3$ | $3^2$ | $3^1$ | $3^0$ |
|---|---|---|---|
| 0 | 0 | 0 | 0 |

Each time we add 1 more to the unit's position $3^0$, it will advance that dial by one until it reaches 1. One more and it will turn that dial around to (-1) and advance by one the value of the next position ($3^1$). Hence, counting on our modified base 3 counter will proceed thusly:

| **Base 10** | **Base 3 Counter** | **Base 10** | **Base 3 Counter** |
|---|---|---|---|
| 0 | 0 0 0 0 | 8 | 0 1 0 -1 |
| 1 | 0 0 0 1 | 9 | 0 1 0 0 |
| 2 | 0 0 1 -1 | 10 | 0 1 0 1 |
| 3 | 0 0 1 0 | 11 | 0 1 1 -1 |
| 4 | 0 0 1 1 | 12 | 0 1 1 0 |
| 5 | 0 1 -1 -1 | 13 | 0 1 1 1 |
| 6 | 0 1 -1 0 | 14 | 1 -1 -1 -1 |
| 7 | 0 1 -1 1 | 15 | 1 -1 -1 0 |

Interestingly, the numerals in each place-value position tell us exactly which weights to use and on which pan to place them in order to effect the desired weighing. A most satisfying and conclusive model of mathematics applied to a real-world problem. The mystery is solved. And our appreciation of the beauty of mathematics is further reinforced.

## Teaching Implications

Until now, we approached the extended problem of *Why powers of 3?* automatically utilizing the full knowledge base at our command – in this instance, drawing upon information that a high school student has yet to learn. Namely, our realization that for modular arithmetic, elements from the same residue class can perform similarly in some instances.

The question now becomes what to do with the solution to this extended problem. Should it be shared with students? Should we merely set it aside as an enrichment exercise which added to our professional knowledge and do nothing more with it? Or should we think further about the problem and how it might be shared with selected student audiences?

Those choices will depend upon individual classroom contexts, including the mathematical capabilities of students and the circumstances under which the extended problem might arise. Setting those aside for the moment, let's look further into related pedagogical questions.

***Is the topic worth sharing?*** What we discovered in modifying the base 3 numeration system to fit the givens of the balance problem was a mathematical model for describing how to place the weights to weigh various amounts. This outcome seems to fall under the Standards of Mathematical Practice (CCSS. MP4) Model with mathematics. This standard says, in part:

> Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace ... They [high school students] routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Certainly, the use of base 3 to model the system of weighing on a two-pan balance appears to fit the standard intent, as well as the ensuing need to revise that model in terms of the problem requirements. That said, the primary value of the extended problem occurs when one or more students constructs the solution themselves rather than observing a teacher presentation. For most students, we would be happy with the discovered reasoning that each individual weight for the balance is used in exactly three different ways. A further consideration is whether you have a follow up activity that will engage students in using the information learned in some meaningful way.

***Making the solution discovery accessible.*** An alternate solution pathway that involves only mathematical ideas available to an Algebra II class is as follows (following the polynomial functions chapter). Examine the second column from Table 1, where we can note that the weight-combination expressions resemble the expanded form for a place value numeral in base 3. However, the coefficients for the powers of 3 are {0, 1, -1} rather than the expected {0, 1, 2}.

This observation leads to the question of whether one can write a base three numeral with {0, 1, -1}. Since weighing is a form of 'counting,' we envision a digital counter with those three numerals on each dial and write out examples of how it will work. This action produces an illustration similar to the one which this article provides. However, we readily stipulate that such a discovery would be a stretch for any but the best students, and even then, might require a hint or two.

Alternatively, this more accessible form of a solution could be shared with a mathematics club or used to guide the discovery of selected students whose interest persists in wanting to know why the mathematics works as it does. Either audience should be able to follow the logic involved.

## Conclusion

The author's primary goal for writing this article is to provide an illustration to the teacher reader of why we choose this profession for our life's work. There is a unique quality about mathematics as a field of study that produces rare moments of beauty when one achieves an insight into how things work – with mathematics as the tool for investigation and discovery. The resulting exhilaration of sudden understanding, where before there was a mystery, is something which is difficult to explain, but known so very well by those who experience it. We hope that you have enjoyed one more of these moments.

But before you put this problem away for good, you might wish to ponder whether the idea of negative digits can be used with other bases. For example, can you count in base 4 with {-1, 0, 1, 2}? Could you use {-2, -1, 0, 1, 2} for a numeration system in base 5?

## References


Common Core State Standards Initiative (CCSSI) (2010). *Standards for Mathematical Practice*. Available online at `www.corestandards.org`.

Meiring, S. (Fall 2020). Stella and the Stunners: The Mathematical Legacy of Rudd Crawford. *Ohio Journal of School Mathematics 87*, 49–57.

Howe, R. (ed.). (2006). *The Quotable Teacher*. Guilford, Connecticut: The Lyons Press.



**Steven Meiring** began his education career in Indiana in 1965, teaching junior high school (all subjects) between earning his bachelor's and master's degrees. After eight years of teaching high school and college mathematics, he came to Ohio to serve 26 years as a mathematics consultant for the Ohio Department of Education. During that tenure, he served on the prestigious Mathematical Sciences Education Board of the NRC and the American Junior High School Examination committee of the Mathematical Association of America.