Produce Basket: The Development of PB module for improving fraction learning

Debe Adams, Terri Bucci, Lee McEwan
The Ohio State University—Mansfield
Kevin Reinthal
Lucas Local Schools

Abstract: *Formal notation of fractions is a critical stumbling block for students, impeding their progress to acquire flexible number sense beyond integers and greatly impacting their success in algebra. Conflicting and vague definitions of fractions are a major cause of confusion and frustration among students and their teachers. Although several authors have identified this lack of clarity and have attempted to help teachers understand fractions better, a successful curriculum unit has not appeared that teachers and students can both use to make the meaning of fractions mathematically precise. To address this lack, a team consisting of a veteran fourth-grade teacher, a professional development leader/university instructor, a math educator, and a mathematician developed an experiential learning module called Produce Basket (PB) for elementary grades fraction learning. The module is a graduated set of games and activities around the metaphor of a basket (standing for "whole") that enables teachers to scaffold the precise definition of a fraction. The aim of this article is to describe the PB learning module with its roots in Bob Moses and the Algebra Project. Additionally, this article will discuss the theory of action that underpins the design of the module.*

Keywords: *fractions, teacher preparation, Algebra Project*

Introduction

Learning fractions takes several years and requires exposure to a rich mixture of metaphors, problem contexts, and representations. The difficulty in teaching rational numbers in the elementary grades has long been known and documented. For example, the National Assessment of Educational Progress reported in 2017 that less than one third (32%) of fourth grade students could correctly decide if a given fraction is less than, equal to, or greater than one-half. Yet the centrality of fractions as a prerequisite for algebra has been noted for decades, and many proposals for improving the teaching and learning of this topic have been advanced without obvious success (Lamon, 2006; Wu, 1999).

There is an aspect of fraction learning that is mysteriously overlooked in the many developed avenues of instruction. Namely, how can students learn to connect formal fraction notation with the intrinsic meanings of the integers that compose it? To elaborate: can a particular unit of games and activities designed around a carefully constructed metaphor help students connect the integers in a fraction to things they can count, and does learning this metaphor improve their overall grasp of fractions?

A perhaps well-worn story illustrates the dilemma: The McDonald's Quarter-Pounder is a famous product introduced over 50 years ago that is known by its name, which references a fraction. The following anecdote was reported by CBC radio, among other sources:

> A&W decided [to] give the Quarter Pounder some competition in the 1980s. So it introduced the "Third-of-a-Pound Burger." It was priced the same as the Quarter Pounder but with a third of a pound of beef, instead of just a quarter pound. It even outperformed the Quarter Pounder in taste tests. But nobody bought it.

Why?

> More than half the people in the focus groups questioned the price of the third-pounder. They wanted to know why they should have to pay the same price for a third of a pound as they did for a quarter pound at McDonald's. They said A&W was overcharging them. You're ripping them off. People genuinely thought a third of a pound was less than a quarter pound. Because 3 was less than 4.

An important goal of our intervention is helping students develop a useful answer to this question: *What does the denominator of a fraction count*?

Many different modalities of proportional thinking are needed to gain a solid understanding of fractions, such as ratios, measurement, quotients, and stretching/shrinking, in addition to part-whole interpretations (Kieran, 1976; Kieran, 1980). It is appropriate for elementary students to begin their encounter with fractions in informal settings, with little attention applied to notation or formal structure. In analogy with the scientific method, this is the "data collecting phase" (Wu, 1999), which should later become the basis for a rigorous formulation ("theory phase"). But the formal mathematical target is frequently obscured or never achieved. In part, the emphasis on the variety of interpretations of a fraction causes the relative simplicity of its definition to be overshadowed or lost. Wu points to this fundamental problem: "The concept of a fraction is never clearly defined in all of K–12." The explicit goal of Produce Basket is to address this lack, by providing a scaffold from easily accessible experiences to the mathematically precise notion of a fraction and its efficient symbolic expression.

Theoretical Framework / Literature Review

Moses' Theory of Mathematics Pedagogy

Robert ("Bob") Moses articulated his ideas about mathematics pedagogy, as well as his beliefs about the relevance of mathematics to advancing civil rights, in his 2001 book *Radical Equations*. His theories have been embodied in several curriculum projects developed for and by the national program Moses founded, called the Algebra Project.

Moses describes his approach as a variation on experiential learning. The Moses "5 Step Curriculum Process" is reflective of Kolb's experiential learning cycle (Kolb, 1984) but is also grounded in the language-based view of mathematics of the analytic philosopher W. Van Quine. This view may be briefly summarized as saying that mathematics is a formalization of metaphors stemming from experience. Moses derives the idea that to gain mathematical understanding requires students to travel a path from a meaningful experience to a formal symbolization of the elements ("features") of the experience. Crucial to this process is that the experience connect with the student's own culture, or at least be accessible to the student without any technical assumptions.

The Kolb model is usually depicted as a circle, with "experience" taking place at 12 o'clock, "reflection" at 3:00, "abstraction" at 6:00, and "application" (apply learning to new situations) at 9:00. The Moses 5 Step Process is an interpolation of the first three positions in the Kolb model. Students first take part in a structured experience. In the next two steps the students Draw and Write about the experience (informal description). These steps engage the students' imagination and natural-language skills. In the fourth step ("Feature Talk"), students analyze their own and each other's descriptions to extract essential features of the experience. In the final step students construct symbols and symbolic expressions to render their feature sentences into formal symbols. (This mode of teaching mathematics typically requires development of a new skill set by the teacher.)

The Algebra Project originated in Cambridge and Boston, MA, during the 1980s and expanded in the 1990s to over 100 middle schools in low-income communities. Research on several schools in Cambridge, Jackson, MS, and San Francisco showed that the Algebra Project graduates enrolled in and passed Algebra I and Geometry courses at about twice the rate of their peers from non-Algebra Project schools in the same districts (Davis & West, 2000). In the 2000s, several mathematicians became involved in creating instructional modules and testing their implementation in high schools in Jackson MS, Petersburg VA, San Francisco, and later in Miami, Los Angeles, Mansfield OH, Eldorado IL, and Ypsilanti MI. Most of the students entered the project performing in the lowest quartile in math on state mandated tests. Research supported by the National Science Foundation found that Algebra Project students performed significantly better on state tests than similar groups of non-Algebra Project students in the same schools. In a study of five high schools, they had higher rates of high school graduation than a matched comparison group, especially if they participated in the project for at least two years (West, 2010; Moses, 2014).

Description and Origin of Produce Basket

The proximate motivation for building a set of activities eventually labeled "Produce Basket" is frustration with student perplexity regarding conventional symbolism for fractions. The mathematician's definition of a fraction is particularly impenetrable to elementary (and older) students:

> A fraction is an ordered pair of integers (a, b) subject to a condition ("b can't be zero") and an equivalence relation: "(a, b) is equivalent to (c, d) if $ad = bc$."

Traditional mathematics pedagogy attempts to infuse students with these statements and their meaning, typically via repetition. This leads students to regard mathematics as a game of symbols with no connection to the physical world.

The definition of a fraction draws from the Latin "fractus", meaning broken, and in ordinary language is always less than one. Fraction definitions reflect this misunderstanding. Here are some random examples chosen from the web: "A fraction represents a part of a whole." "A small or tiny part, e.g., a fraction of a second." "When we divide a whole into equal parts, each part is a fraction of the whole." From Oxford Languages Dictionary: "a numerical quantity that is not a whole number (e.g., $\frac{1}{2}$, 0.5)." A student is much more likely to retain these definitions than a formal definition.

Background and Rationale

Operations with fractions is one of the leading struggles that students (and adults) have with mathematics. The confusion of fractions begins early in elementary education when students are given the definition of fraction as "part of a whole" which implies that a fraction is always less than one. While students are discovering how to work with "parts of a whole," they are given the new and unusual notation of a fraction where their familiar $0 - 9$ digits suddenly mean something other than counting how many. The new notation of $\frac{a}{b}$ is an abstract representation with no single concrete basis; instead, there are numerous descriptions that appear mutually incompatible (Wu, 1999). To add to the struggle, many teachers lack confidence and are unable to offer a better pathway because of their own experiences with fractions. A better definition of fraction is *a number that tells us "how many" and "what kind."* The parts of a fraction are numerator, meaning "to count" (how many), and denominator, meaning "name" (what kind). Before students are given the traditional mathematical language and notation, they should have experience with identifying **how many** and **what kind**.

Description of Intervention

Produce Basket (the name stands as a metaphor for "whole") is a teacher-developed approach to grounding standard fraction notation in a set of experiences that allow students to discuss and explore the meanings of the inputs to fraction notation. Those inputs are seen in context as answers to different counting problems: How many do I have and how many make a whole? The PB module integrates tactile experience with fractions in a fun zero-prerequisite context with a direct scaffold to the precise definition of a fraction as a number, having a unique position on the number line. Critical to this program are the game/activity roles of the integer components of the fraction, so that their precise mathematical roles in a formal fraction make sense. Students can use PB to readily re-contextualize from $\frac{a}{b}$ to a sense-making context for both integers, not just the numerator.

We have not seen a true parallel to this approach within any of the standard curricula that we are familiar with, either in traditional texts such as Go Math, or innovative/explorative approaches such as Investigations 3.

The Produce Basket is a set of core activities created and used by Kevin Reinthal, a veteran elementary school teacher. Kevin wanted to give his students a deeper understanding of fractions than what the standard approach and algorithms provided. This desire led him to tweak and refine these activities over the years as he responded to the students' feedback, engagement, and comprehension of fractions.

These activities build a concrete physical model of fraction concepts and, when strategically implemented, provide students with a path to scaffold this physical representation to the conventional abstract symbolism of fractions.

The four activities of the Produce Basket described below are intentional in sequence and accompanied with deliberate vocabulary and notation. The activities begin with problems that require only counting and are then slowly and purposefully developed into formal mathematics.

Game Play

Purpose

While there is no substitute for experiencing the activities of Produce Basket directly, it is hoped that these brief descriptions give some idea of how the module works. In summary, the purposes of Produce Basket are (1) to give a direct route to a precise definition of a fraction as a number, (2) provide a flexible metaphor to ground fraction operations, and (3) to give students a simple tool to use for interpreting the many complex and confusing situations in which fractions arise.

Game Pieces and Objectives of Play

A set of plastic pieces of different colors and sizes are named. A 1"x 8" plastic rectangle (preferably black, but we used red) is the "basket," which later becomes generalized into 1 whole unit. Yellow pieces, bananas, are 1"x 2" so four bananas will "fill" the basket. In future lessons, bananas will become fourths. Different colored and sized pieces, each with a produce name, represent halves, thirds, fourths, fifths, sixths, eighths, tenths, and twelfths. Once students are familiar with the names, colors, and relative sizes of the pieces an initial exploration of fraction concepts can begin, without the distraction and confusion of standard fraction notation. All the concepts essential to fractions as numbers can be explored without formal notation. Equivalence of fractions begins with tasks such as: Find how many blackberries fill the same amount of the basket as 1 pumpkin? Comparison of fractions are explored by asking: Which fills more of the basket, 5 blueberries or 2 bananas? Computation problems start with a question: If there is a watermelon in my basket, how many blueberries will it take to fill the basket? In this setting, all fraction questions are related directly to counting.

Figure 1: *Game pieces.*

Produce Basket Activities

The Produce Basket Board Game

The Produce Basket Board Game is an introduction to the idea of counting and naming objects that are different sizes individually, but when combined in different multiples may fill equivalent space. By design, the PB Board Game is a variation on many games familiar to most children in which player pieces move around a path according to a throw of a die. It is easy to play while supporting informal learning of the essential elements of the later more-structured activities of Produce Basket. The game allows the students to begin with a concrete experience and discuss the experience using their own language. Mathematical language is not used so students are not weighted with formal notation. This activity provides the students time to become established in thinking about *how many and what kind.* The goal is to gain informal familiarity with PB objects and their properties.

Figure 2: *How to play the Produce Basket Board Game (Source: https://youtu.be/FEKu_0Hk21g).*

Fill the Baskets

Fill the Baskets, the central game/activity of PB, is a a board game that helps the students begin to transition to standard fraction notation. In this game, emphasis is placed on “how many” and “what kind” using the features of two rolled dice of different sizes. The first die gives “how many” and the second die determines “what kind.” In the beginning the second die uses colors (corresponding to the fruit from the PB board game) instead of numbers. The dies themselves are different in size to emphasize their different purpose.

Figure 3: *Produce Basket Die.*

The "board" for this game (shown in miniature below) is already familiar to many teachers: it is a sheet of empty rectangles which can hold game tiles. Each rectangle represents a whole basket.

Figure 4: *Empty rectangles.*

In addition to the game board, students need one set of blank rainbow fraction tiles per student and two dice as described above. Player 1 starts by rolling the small die. The number on that roll (1-6) determines the number of pieces a player will get to place in that turn. Player 1 then rolls the larger die (pink, orange, yellow, light blue, dark blue, black). The color on that roll determines the kind of piece the student will get to place. If a player rolls a “4” on the small die and a “yellow” on the large die, the player selects 4 banana pieces and may place them together, separately, or in any combination on any of the baskets where there is room. Pieces should be placed from left to right in a basket with no spaces left between pieces. Players must place all the pieces that they can during a roll. If a player is unable to place all or some of the pieces, the pieces for which there is no space are returned to the player’s set, and the turn is over. Play continues until the baskets are all filled, there is no piece that will fill any unfilled baskets, or the players go through a complete cycle of turns with none of the players being able to place a piece.

A critical transition is facilitated by a teacher-led discussion of using numbers to represent the kinds of produce. For example, students have learned by playing Produce Basket Board Game that six plums (purple tiles) fill a basket, so it is natural, and students will typically volunteer this, to represent a plum by the numeral 6. If students do not suggest this numbering scheme, then teacher questions such as "Is there a convenient way to just use numbers?" is necessary and usually sufficient. After this is agreed to, when a student rolls a 5 for how many and a 6 for what kind, it means five plums. Throughout this transition students will move from their own language of “five plums” to a shorter notation, perhaps “5 Ps.” As they progress, students will interchange “5 plums” and “5 6s” or “5 6”, using notation of different levels of abstractness. The motivation of the eventual transition to using only numerals is that numbers are efficient and have a direct meaning that has been acquired by the students. The final move to formal fraction notation ($\frac{5}{6}$) then becomes natural. This process is skillfully guided by the teacher and is key to student readiness.

Figure 5: *How to play the Fill the Baskets (Source: https://youtu.be/yUBu9KDzico).*

Transitioning to Formal Fractions

To elaborate this crucial step in the PB curriculum: After playing Fill the Baskets, the process of transitioning to formal fraction notation begins. Fill the Baskets is purposefully designed to isolate two pieces of information with the roll of two unique, differently sized dice: The small die plays the role of the numerator – how many pieces, and the large die represents the denominator – what kind of piece. The transition phase begins during the course of subsequent activities. Suppose, for instance, that following a session of Fill the Baskets play, the teacher asks the students to find all possible sets of small die/large die roles that would exactly fill one basket. One possible student response is four (on the small die), banana (yellow on the large die). That response can be recorded as: "Four Bananas"

The students then need to be tasked with discovering simpler ways of expressing that information, using more efficient symbolism. Adhering to class suggestions and consensus, the symbolism can evolve during subsequent lessons. It might become "4B" or "4 Bn", as the students notice the confusion resulting from bananas and blueberries starting with the same letter. Eventually either children or the teacher suggests the possibility of numbering the produce pieces in some sort of logical way.

Figure 6: *Kevin explains how to make the transition with students from the Produce Basket to formal fraction notation (Source: https://youtu.be/uLl_qu5XDjU).*

More Activities

Trader's Game

Trader's Game is a game that uses a game board configuration that is often seen in standard textbook figures to illustrate equivalent fractions. Students roll a die to determine the number of small pieces they may place on their board. When possible, students may trade smaller pieces for an equivalent larger piece. Students may use a combination of produce metaphor language or standard fraction terminology such as "two bananas trade for one watermelon" or "2 fours equal 1 two" or "$\frac{2}{4} = \frac{1}{2}$." The teacher is guiding the language/notation at the pace needed by the students and asking questions to address grade-level benchmarks.

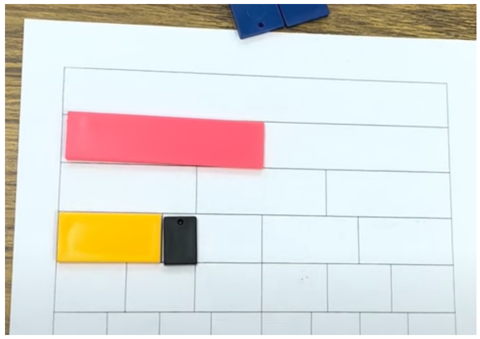

Figure 7: *How to play the Trader's Game (Source: https://youtu.be/Auvid0BX_94).*

Garden Pest

Garden Pest is a board game that transitions the produce pieces onto a number line. As students compete to sink their opponents' insect pest, they determine how many and what kind of lengths (same as produce in previous games) they want to jump to "hit" the opposing pest. Players roll a die to determine "how many" and then they freely choose "what kind" they want to use. Example: If a player rolls a 4, they may choose to move $\frac{4}{3}$, $\frac{4}{12}$, etc. Students strategically use equivalent fractions, become fluent with fractions greater than one, and spontaneously see fractions as numbers on a number line.

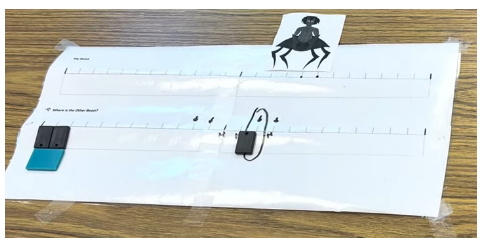

Figure 8: *How to play the Garden Pest (Source: https://youtu.be/wZzMBObcYuU).*

Summary and Final Thoughts

The PB model has been implemented in several classrooms over the past 5 years. It is also being used for preservice teachers at Ohio State University Mansfield. In-service teachers who have adopted Produce Basket have been strong advocates for continued use and development of the PB module in their schools and provide us with steady demand for training. More research is needed to document the effectiveness of this module and approach and provide more evidence that a metaphor for fraction learning built along the lines of Moses' experiential philosophy can lead to a mathematically precise definition of a critical concept and help students learn.

It is useful to note that the materials that are used in Produce Basket are inexpensive and easily obtained. All supporting documents, such as game boards, rules, and teaching guides and goals are available without cost. Physical materials, namely produce tile and dice, are available online or adaptable from existing resources.

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Debe Adams (adams.2167@osu.edu) teaches college-level math courses including courses for pre-service teachers at The Ohio State University, Mansfield. In 2009, she began teaching The Algebra Project pedagogy in the high school classroom and currently provides professional development grounded in the pedagogy of The Algebra Project for K-12 in-service math teachers through OSU Mansfield's Math Literacy Initiative (MLI).

Terri Bucci (bucci.5@osu.edu) is an associate professor and researcher in mathematics education. Terri's research interests focus on mathematics education and international teacher education, with a focus on the professional preparation of primary grades teachers in Haiti. Dr. Bucci has worked collaboratively with Haitian and US colleagues for many years and is currently involved in a collaborative project which connects the human, intellectual, and fiscal resources between US faculty of education and Haitian colleagues.

Lee McEwan (mcewan@math.ohio-state.edu) is professor emeritus of mathematics at Ohio State University's Mansfield Campus. Lee's areas of expertise include algebraic geometry and the topology of algebraic singularities. A long-time advocate of the Algebra Project, Lee co-founded the Mathematics Literacy Initiative (MLI) at OSU-Mansfield with Terri Bucci and collaborated with Kevin Reinthal to develop Produce Basket.

Kevin Reinthal (koreinthal@gmail.com) taught mathematics in fourth grade for 30 years in the Lucas Local Schools in Loudonville, Ohio. He is the lead developer of Produce Basket. As a Math Teacher Leader, Kevin has worked with dozens of in-service teachers to learn and adapt Produce Basket to their classrooms. He led the intervention in Ms. Adams' classroom.

A Note about the Authors

Because of the distribution of strengths of our team we opted to list the authors in alphabetic order. The context of this work is the Mathematics Literacy Initiative at the Ohio State University, a multi-year effort led by Dr. Terri Bucci. The MLI grew out of work that Bucci and McEwan did with Bob Moses and the Algebra Project starting in 2009. Kevin Reinthal is a veteran elementary school teacher who undertook professional development training through the MLI. He identified the issues with teaching fractions addressed in this work and is the primary developer of Produce Basket. He co-taught the intervention for this research. Debe Adams has a background with the Algebra Project also dating to 2009, having taught Algebra Project aligned high school mathematics classes under a 5-year NSF grant in Illinois before coming to OSU as a PD specialist with the MLI. She is the primary instructor for the courses in this intervention and aided in data collection and analysis. Lee McEwan is a retired mathematician at OSU who has worked with the MLI and the Algebra Project for professional development of K-12 mathematics teachers and served as Mr. Reinthal's partner in developing Produce Basket.