A New Divisibility Test for a Class of Integers: The Frey-Hammett Divisibility Test

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Abstract: *The author discusses a recently-discovered divisibility test for the infinite class of integers ending in 1, 3, 7, or 9. He explains how to apply the divisibility test, provides examples, includes an explanation (proof) of why the test works, and provides some ideas to try in the classroom. The article ends with a method for extending the divisibility test to all integers.* ***Keywords:*** *Divisibility tests, number theory, sense-making, proof*

A Brief Overview of Divisibility Tests

Many teachers share with their students some of the common divisibility tests for knowing whether or not a certain integer is evenly divisible by another integer. Of course, we can easily determine if an integer is (evenly) divisible by 2 or not depending on whether its units digit is a $0, 2, 4, 6$, or 8. Similarly, an integer is divisible by 5 if and only if its units digit is a 0 or 5. Perhaps the other divisibility tests that are most familiar are those for 3 and for 9. An integer is divisible by 3 if and only if the sum of its digits is also divisible by 3, and an integer is divisible by 9 if and only if the sum of its digits is also divisible by 9.

Of course there are other lesser-known divisibility tests. An integer is divisible by 11 if and only if the alternating sum of its digits is divisible by 11. For example, to check to see whether the number $14{,}517{,}262{,}925$ is divisible by 11, we compute the alternating sum of its digits:

$$1 - 4 + 5 - 1 + 7 - 2 + 6 - 2 + 9 - 2 + 5 = 22,$$

and since 22 is divisible by 11 then so also is $14{,}517{,}262{,}925$ divisible by 11. And a very useful theorem tells us that if two numbers a and b are relatively prime (they have no common factors other than 1), then an integer n is divisible by the product ab if and only if n is divisible by a and n is also divisible by b. I will call this "Fact 1" because we will be referencing it later in this article.

Fact 1

If a and b are relatively prime, then an integer n is divisible by the product ab if and only if n is divisible by both a and by b.

For example, is the number 104 divisible by 6? Well, $6 = 2 \times 3$, so we can check to see if 104 is divisible by both 2 and by 3. 104 is divisible by 2 (since its units digit is 4), but it is not divisible by 3 (since $1 + 0 + 4 = 5$ and 5 is not divisible by 3). So the integer 104 is not divisible by 6. Similarly, we can see that the number $22{,}455$ is divisible by 15 ($15 = 3 \times 5$) by seeing that $22{,}455$ is divisible by both 3 and by 5.

But a recent result on divisibility tests that was written about in the January 2021 issue of *The College Mathematics Journal* (Frey and Hammett, 2021) is quite interesting. The authors and discoverers of the test, Darrin Frey and Adam Hammett of Cedarville University, provide a very general divisibility test that works for an entire class of integers, namely any integer that is relatively prime to 10 (so it is relatively prime to both 2 and 5). This means that it works for any integer that ends in $1, 3, 7$, or 9, since the only integers that are not relatively prime to either 2 or 5 are those ending in $0, 2, 4, 5, 6$, or 8.

I will briefly explain how to apply the test, which I will call the Frey-Hammett Divisibility Test in recognition of the discoverers of it (although I have not seen the test called this anywhere else yet). Then we will look at two examples of how the test works, before discussing the reason why the test works (and provide a proof). Then I will provide two more examples for you to try (with hints to make sure you are applying the test correctly) and which may be used in the classes you teach. Finally, we will end with a simple extension of the Frey-Hammett Divisibility Test that allows us to check a number for divisibility by any other number (not just those ending in 1, 3, 7, or 9).

How to Apply the Frey-Hammett Divisibility Test

We first note that the class of integers to which this test can be applied are those integers ending in $1, 3, 7$, or 9. That is, we can test to see whether any integer N can be divided by any other integer p provided p ends in a $1, 3, 7$, or 9.

We will use the fact that any integer p ending in 1 or 9 can be written as $p = 10k \pm 1$. And any integer p ending in 3 or 7 will, upon multiplication by the number 3, end in either 9 or 1. So in this case, we can write $3p = 10k \pm 1$.

So the Frey-Hammett Divisibility Test works like this: Given an integer N that we wish to test whether or not it is divisible by an integer p (where p ends in $1, 3, 7$, or 9), we first determine the value of k for which either $p = 10k \pm 1$ or $3p = 10k \pm 1$. We will also let a represent the units digit of the integer N.

The Frey-Hammett Divisibility Test states that N is divisible by p if and only if the integer $\frac{N-a}{10} \mp (a)(k)$ is divisible by p. Note that the first part of this expression, $\frac{N-a}{10}$, is simply the integer obtained by truncating the units digit from N. For example, if $N = 56{,}378$ then $\frac{N-a}{10} = \frac{56{,}378-8}{10} = 5{,}637$. Note also that between the two terms of the expression we have the $\mp$ symbol, which indicates that we use the opposite operation here than the operation that was used in determining the value of k from either $p = 10k \pm 1$ or $3p = 10k \pm 1$.

Example of the Frey-Hammett Divisibility Test

Suppose we wish to determine whether or not the integer $N = 2{,}813{,}984$ is divisible by $p = 47$. Using common notation for divisibility, we will be checking to see if $47|2{,}813{,}984$. First, we note that the units digit of N is 4, so we have that $a = 4$. Next, we note that $p = 47$ does not end in a 1 or a 9, but $3p = 141$ does end in a 1. In particular, we see that $3p = 10(14) + 1$, so our value for k is 14. We also see that the operation of addition is being used in the expression $10(14) + 1$, so the operation of subtraction will be used in the truncation step.

Now that we know that $a = 4$, $k = 14$, and we will be using the operation of subtraction, the test states that $47|2{,}813{,}984$ if and only if $47|[281{,}398 - (4)(14)]$, i.e., $47|281{,}342$.

But to check to see if $47|281{,}342$ we can simply repeat the process, using the new value for a of 2. So $47|281{,}342$ if and only if $47|[28{,}134 - (2)(14)]$, i.e., $47|28{,}106$.

- Repeating again using a value for a of 6: $47|28{,}106$ if and only if $47|[2{,}810 - (6)(14)]$, i.e., $47|2{,}726$.
- Repeating again using a value for a of 6: $47|2{,}726$ if and only if $47|[272 - (6)(14)]$, i.e., $47|188$.
- Perhaps at this point we see that it is true that 47 divides 188, but if not then we can apply the test again using an a-value of 8: $47|188$ if and only if $47|[18 - (8)(14)]$, i.e., $47| - 94$, and of course 47 divides into -94 evenly.

We can observe that by truncating the units digit at each step, we are reducing the number to be tested by a factor of approximately 10 on each step. So if $N = 569{,}473{,}211$ then we are starting with a 9-digit number and after 7 iterations of our test we should (in general) be down to a 2-digit number that is being tested.

Another Example of the Frey-Hammett Divisibility Test

Let's see if the integer 360,206,132 is divisible by 219. So $N = 360{,}206{,}132$ has 9-digits, so we should expect to do six iterations of the test to get down to a 3-digit number (which is the number of digits in p) that we are testing. We note that $p = 219$ can be expressed as $p = 10(22) - 1$, so $k = 22$. We also observe that we are using the operation of subtraction in the expression $10(22) - 1$, so we will be using the operation of addition in the truncation step.

The Frey-Hammett Divisibility Test states that $219|360{,}206{,}132$ if and only if $219|[36{,}020{,}613 + (2)(22)]$, i.e., $219|36{,}020{,}657$.

- A second iteration of the test says that $219|36{,}020{,}657$ if and only if $219|[3{,}602{,}065 + (7)(22)]$, i.e., $219|3{,}602{,}219$.
- A third iteration shows that $219|3{,}602{,}219$ if and only if $219|[360{,}221+(9)(22)]$, i.e., $219|360{,}419$.
- A fourth iteration gives us $219|360{,}419$ if and only if $219|[36{,}041 + (9)(22)]$, i.e., $219|36{,}239$.
- A fifth iteration: $219|36{,}239$ if and only if $219|[3{,}623 + (9)(22)]$, i.e., $219|3{,}821$.
- A sixth iteration: $219|3{,}821$ if and only if $219|[382 + (1)(22)]$, i.e., $219|404$.

We can see clearly that 219 does not divide 404, so 219 does not divide 360,206,132.

We also note that 219 is not a prime number ($219 = 3 \times 73$). It does, however, end in a $1, 3, 7$, or 9 and so the Frey-Hammett Divisibility Test applies to it.

A Proof of the Frey-Hammett Divisibility Test

A formal statement of the test could be as follows.

Theorem. *Let N and p be positive integers, with p relatively prime to 10. Let k be such that either $p = 10k \pm 1$ or $3p = 10k \pm 1$. Then p divides N if and only if p divides $\frac{N-a}{10} \mp (a)(k)$, where a is the units digit of N.*

Proof. Suppose that $p| \left[\frac{N-a}{10} \mp (a)(k)\right]$. This is true if and only if $p|10 \left[\frac{N-a}{10} \mp (a)(k)\right]$ since we know that p does not divide 10.

Distributing the 10, we see that this means the original divisibility statement is true if and only if $p|\,[N - a \mp 10(a)(k)]$.

Factoring out the a from the last two terms in the expression we get that the original divisibility statement is true if and only if $p|\,[N - a(1 \pm 10k)]$, where the $\mp$ switched to $\pm$ since we factored the negative sign out with the a.

But we know that the expression $10k \pm 1$ is equal to either p or $3p$. So that means that the original divisibility statement is true if and only if $p|\,[N - a(p)]$ or $p|\,[N - a(3p)]$. In either case, we see that p evenly divides the second term in each of those expressions, which means that p must also divide N. Note that we are using here the fact that if $a|(b+c)$ and $a|b$, then it must be true that $a|c$ also. Thus, $p|\left[\frac{N-a}{10} \mp (a)(k)\right]$ if and only if $p|N$, and the theorem is proved. □

We will be referring back to the fact used at the end of this proof later, so let's call it "Fact 2."

Fact 2

If $a|(b+c)$ and $a|b$, then $a|c$ also.

It is worth noting that the proof does not require that p be a prime number, merely that p be relatively prime to 10 so that p does not divide 10. So as long as p has a units digit of $1, 3, 7$, or 9 then the Theorem holds true.

Try These Examples

Example 1: Is 25,783 divisible by 59? How many iterations will it (likely) take to reduce then number that you are checking to a 2-digit number?

- *Hint at solution*: You should determine that $k = 6$ (since $59 = 10(6) - 1$), and that you will be using the operation of addition in the truncation steps. After three iterations, you should obtain that $59|59$, so 59 does divide evenly into 25,783.

Example 2: Is $8{,}066{,}562$ divisible by 81? How many iterations will it (likely) take to reduce the number that you are checking to a 2-digit number?

- *Hint at solution*: You should determine that $k = 8$ and that you will be using the operation of subtraction in the truncation steps. After five iterations, you should obtain that $81|69$, which is not true. So 81 does not evenly divide into $8{,}066{,}562$.

A Simple Extension of the Frey-Hammett Divisibility Test

We've seen that the Frey-Hammett Divisibility Test can be applied to any integer p that ends in $1, 3, 7$, or 9. We can, in fact, extend the test to the remaining integers—those ending in $0, 2, 4, 5, 6$, or 8—by simply using Fact 1: if two integers are relatively prime, then their product will divide a number if and only if each of the two integers divide that number.

If the number p that we are using does not end in $1, 3, 7$, or 9, then it either ends in $2, 4, 6$, or 8; or it ends in 5; or it ends in 0. We will first look at the case where p ends in $2, 4, 6$, or 8. So p is divisible by 2, and in fact can be divided by some power of 2. We will factor p into the highest power of 2 that divides into it, multiplied by the remaining factor of p. In other words, we will write $p = 2^q \cdot M$, with M ending in $1, 3, 7$, or 9. For example, if $p = 472$ then we rewrite it as $472 = 8 \cdot 59 = 2^3 \cdot 59$. Furthermore, 2^q and M will be relatively prime to one another.

To check if a power of 2 divides a number, we need only to look at the last few digits. We know that 2 divides a number only if the last digit is $0, 2, 4, 6$, or 8. This is because 2 divides 10, and every number can be written as a multiple of 10 plus its units digit. For example, we know that 2 divides 78 because $78 = 70 + 8 = (7)(10) + 8$, and since 2 divides $(7)(10)$ then it will divide 78 if and only if it divides 8 (we are using Fact 2 here). Similarly, $2^2 = 4$ will divide a number only if the last two digits are divisible by 4. This is because 4 divides 100, and so will divide every multiple of 100. For example, we can easily see that 4 divides 732 because $732 = (7)(100) + 32$, and since 4 divides $(7)(100)$ then 4 will divide 732 if and only if 4 divides 32 (which it does). In similar fashion, since 8 divides 1000 we need only check to see if 8 divides the last three digits of an integer to know if 8 divides the integer itself; to check to see if 16 divides an integer we need only check to see if 16 divides the last four digits of the integer; and so on.

Suppose we want to check to see if $N = 2{,}924{,}984$ is divisible by 472. We know that $472 = 2^3 \cdot 59$, with 2^3 and 59 relatively prime. So 472 will divide 2,924,984 if and only if both 2^3 and 59 divide 2,924,984. We can check that $2^3 = 8$ divides 2,924,984 since 8 divides 984. We can also use the Frey-Hammett Divisibility Test to confirm that 59 divides 2,924,984, so $472 = 2^3 \cdot 59$ does divide 2,924,984.

In similar fashion, we can check to see if an integer p that ends in 5 will divide into another integer by rewriting p as the product of its highest power of 5 multiplied by the remaining factor of p. We then use the fact that 5 divides an integer if and only if its last digit is divisible by 5 (since 5 divides 10 and using the same reasoning as before); that $5^2 = 25$ divides an integer if and only if its last two digits are divisible by 25 (since 25 divides 100 and using the same reasoning as before); that $125 = 5^3$ divides an integer if and only if its last three digits are divisible by 125 (since 125 divides 1000 and using the same reasoning as before); etc.

So suppose we want to determine if $p = 1{,}825$ divides $N = 177{,}375$. We first rewrite $p = 25 \cdot 73 = 5^2 \cdot 73$, with 5^2 and 73 being relatively prime. We can easily see that $5^2 = 25$ divides 177,375 since 25 divides 75. Then we can use the Frey-Hammett Divisibility Test and determine that 73 does not divide 177,375. Therefore, $p = 1{,}825$ will not divide 177,375 either.

If an integer p ends in 0 then it is divisible by a power of 10. So we rewrite $p = 10^q \cdot M$, with 10^q and M relatively prime, and check for divisibility by both 10^q and by M. For example, does $p = 91{,}000$ divide $N = 17{,}745{,}000$? Rewriting $p = 91{,}000 = 1{,}000 \cdot 91 = 10^3 \cdot 91$, we first see that 17,745,000 is divisible by 1,000. We then use the Frey-Hammett Divisibility Test to see that 17,745,000 is also divisible by 91. So $p = 91{,}000$ does divide $N = 17{,}745{,}000$.

Finally, these ideas can all be further extended for cases such as $p = 680 = 2^3 \cdot 5 \cdot 17$, by checking for divisibility by 2^3 and by 5 and by 17, all separately.

References

Frey, D. and Hammett, A. (2021). An Infinite Family of Divisibility Tests. *The College Mathematics Journal. 52*(1), 2–10.

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