Investigation and Discovery*

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Abstract: *Investigation and discovery are how mathematics is applied outside the classroom. Richard Skemp, author of The Psychology of Learning Mathematics, says that gradually equipping students with the analytical skills to address mathematical situations without their aid is one of three tasks for mathematics teachers. This ability involves a different skill set than understanding the mathematics for solutions. The self-talk required to explore a non-routine situation to determine how to get started mimics questions the teacher uses to guide students through assisted learning. Discovery is the creative process of modifying existing schemas to accommodate the new situation. Whereas problem-solving is considered by students as narrowing to an answer, investigation and discovery are expansive to new learning.*

Keywords: *problem solving, inquiry*

> We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on. So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work.—**Richard Philips Feynman**, *Nobel Lecture, 1966*.

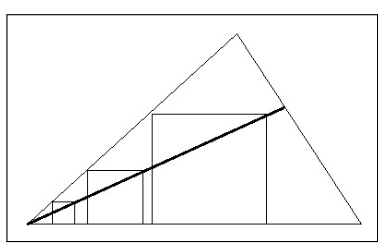
Introduction

In George Pólya's classic *How to Solve It*, he discusses the power of heuristics to solve seemingly difficult problems effectively. This is one such problem he uses:

> Inscribe (construct) a square in a given triangle. Two vertices of the square should be on the base of the triangle, the two other vertices of the square on the other sides of the triangle, one on each.

What begins as a formidable problem is vanquished by the heuristic *Simplify a condition*. Visualizing squares that meet three problem conditions, but not four, illustrates a pattern whereby squares shrink to a point (vertex of the triangle) in one direction and grow to the required solution in the other.

Figure 1: *Task from Pólya's* How to Solve It.



*__Editor's Note__: The following article is the first of three linked articles by this author, published by OJSM over three consecutive issues. ***Mathematics and Thinking***, in the next edition, and ***A Laboratory for Secondary Math*** will follow. The articles present three recommendations for systemic mathematics change across the secondary grades

What Pólya addresses in consummate fashion is how the mathematics works. What he does not address so well is how the problem solver's mind works in creating such an approach in the first place. Earl Hunt says this of Pólya (Sternberg, 1994):

> Pólya offers many illustrations of problems that appear difficult, but can be solved quickly after the author shows that there is a way of looking at the problem that reveals the key underlying relationships. What Pólya did not do was explain how he found an appropriate way to look at a problem *before* he knew the solution.

Hunt goes on to advise that the teaching of mathematics must endeavor to connect student thinking with the process of discovery—rather than focusing exclusively on the elegance of a solution and then expecting the student to emulate a similar brand of thinking.

Raising the Curtain on Discovery

Imagine the following teaching-learning scenario led by a teacher of an algebra II class:

I want you to get with your study partner and try to come up with a solution to this problem in the next few minutes. The problem is ...

> A chicken and a half lays an egg and a half in a day and a half. How many eggs will 12 chickens lay in 12 days?

Resuming the class five minutes later, the teacher asks, *Who wants to share their answer?*

One student holds up his hand and says, *We got 12 eggs*. The teacher responds, *How did you get that?*

The student explains that he and his partner saw the pattern and determined that 12 chickens would lay 12 eggs in 12 days.

Any other possibilities? queries the teacher.

Another student holds up her hand, saying, *We got 144 eggs. We decided we should multiply the given information by* $\frac{2}{3}$ *to get one chicken lays one egg in one day—sort of like the first answer. Then we reasoned that multiplying the time by 12 would increase the number of eggs by 12. But so would multiplying the number of chickens by 12. So we thought the net effect would be* 12×12 *or* 144.

The teacher says, *Hmm ... interesting. I notice that each of your pairs acted on the initial information to try to simplify it. Each arriving at the conjecture that 'one chicken lays one egg in one day.'*

Let's try to go about this yet another way. Suppose this problem arose because we just moved out to the country to try our hand at raising chickens. We approach a neighbor farmer to ask the question of how many eggs we should expect from our flock of chickens.

The farmer smiles and drawls, *You're city folk aren't ya'? Well, here's an answer to your question.* He responds by giving us the same information as that in our problem, and then walks off smiling.

We are back where we started, so we decide to poke around the information a little. We muse *... Suppose we try changing one thing at a time, say ... doubling the number of chickens and note the effect on egg production.* After 15 seconds for students to consider her suggestion, the teacher writes this on the board under the starting information:

3 chickens lay 3 eggs in a day and a half

Suppose instead that we had doubled the length of time, but kept the number of chickens the same:

one and half chickens lay 3 eggs in 3 days

The teacher lets these two statements sink in, before continuing: *Now suppose we doubled both the number of chickens and the number of days at the same time. Then what is the effect produced?* Eventually, she writes: 3 chickens lay 6 eggs in 3 days

Now an impatient student blurts out: *Then, does that make the second answer right? The teacher seems to mull over this question before saying: Well . . . why don't you continue in this way to find out?* Eventually, this chain of statements is written on the board:

A chicken and a half lays an egg and a half in a day and a half.
3 chickens lay 3 eggs in a day and a half.
3 chickens lay 6 eggs in 3 days.
6 chickens lay 24 eggs in 6 days.
12 chickens lay 96 eggs in 12 days.

Until this point, this could have been just a discovery lesson, but now it takes an interesting turn. The teacher asks the class, *How confident are you in this answer (pointing to 96)? After all, it is just one more candidate from among three solution methods.* The class is reluctant to question an answer arrived at through the teacher's direction, even though individually a few may harbor this very question.

Next the teacher asks, *By a show of hands, how many think that I (as the problem solver) immediately knew how to go about solving this question when I first read it?*

After the show of hands, she assures the class that she did not know immediately how to proceed. That the first thing she noted, after reading the question, was that egg production depended upon more than one variable. That prompted her to begin asking herself mentally the same questions that drove the class discussion. She wanted to discern how the variables were behaving relative to one another.

Now, the teacher projects the set of questions she asked previously (written down throughout their interactive dialogue) – questions which no one had paid specific attention to up to this point.

How did each of these questions move along our investigation as it proceeded? For example, why did I have us imagine a setting in which this problem might have arisen?

As they go through each directive question, she calls attention to the thinking that lay behind the search for a solution, rather than the connecting logic of the solution steps. She only does this now because student attention can focus on only so many things at once (limits of working memory).

Eventually she asks, *If these kinds of questions were helpful and you find yourself working alone . . . on the next problem, stuck on a test item, on the job . . . who has to ask these questions to guide your thinking?* The concept of 'self-talk' is introduced as a problem-solving tool.

Following this brief discussion, she says, For the rest of today's discussion, I want you to pay particular attention to each question I ask of the class, how it directs our discussion, and why you think I asked it.

Now the teacher switches gears once more, directing the class' attention back to the original problem. She says, *Are any of you asking yourself, 'Shouldn't there be a formula for handling this kind of question?'*

What follows is a discussion in which the class observes that the original question was a direct variation situation, but one in which the dependent variable e (number of eggs) depended upon two independent variables c (number of chickens) and t (number of days). That direct variation is a situation described by $y = kx$ where k is the constant of variation. However, in this instance, egg production varies directly as the product of the number of chickens and amount of time (confirmed by revisiting the steps in the prior discussion). Therefore, a formula covering this situation might look like $e = k \cdot c \cdot t$.

What must 'k' represent in this formula? the teacher asks. Confirmed is the interpretation that it must be the rate of eggs laid per chicken per day. Now it is a simple matter to fill in the given information to determine the value of k (which is $\frac{2}{3}$). The last step is then to use the formula $e = \frac{2}{3} \cdot c \cdot t$ to arrive at the same answer (96 eggs) by answering the question directly by this means.

After this calculation and confirmation, the teacher says, *Where have you seen a formula like this before—a situation in which the dependent variable varies directly as the product of two independent variables and a constant rate?* (One possible answer supplied by a student or the teacher is $i = P \cdot r \cdot t$ for interest problems.)

Finally, the lesson ends with the following question and directions: *You have the following problem to solve overnight. My additional question for you is, 'What does this problem have to do with the chicken problem?' Both your solution to the problem and answer to my question are to be handed in tomorrow.*

> A wall of 700 yards in length is to be built in 29 days. Twelve men were employed on it for 11 days, completing 220 yards. How many men must be added to complete the wall in the required time?

Also by tomorrow, I want you to read the next lesson of our textbook, which coincidentally happens to be about joint variation.

For the secondary math teacher, there is much to ponder in the solutions of these two non-routine problems in light of Hunt's criticism—of Pólya in particular, and of mathematics learning generally. To be fair, the solutions and criticism should be judged through lenses appropriate to their purpose.

Like most solutions, Pólya's construction takes into account the abilities of the reading audience, omitting many details of the solution, depending on the reader to fill those in himself, once the main idea is presented. He also presumes the analytical skills of the reader are sophisticated enough to appreciate the elegance and brevity of his solution approach as contrasted to other possibilities.

Taken in this context, Pólya's solution rightfully 'fits' the purpose and audience of the publication *How to Solve It*. After all, he did not title his work, *How to Think It*. So, from the lens of *how the mathematics works*, we find Hunt's criticism misplaced.

That said, Hunt, a psychologist, applies a different lens: *how one's thinking occurs in applying what one knows*. A different perspective, appropriate for consideration by educators. Hunt's conclusions are dead-on when applied to mainstream 7-12 educational opportunities: (1) that teaching of mathematics should endeavor to connect student thinking with the process of discovery; and (2) that it is unrealistic to expect students to emulate a brand of thinking, similar to elegant, finished solutions, without having learned first how the process of discovery occurs.

Earl Hunt is not alone in this observation. Richard Skemp, author of *The Psychology of Learning Mathematics*, says this about teaching:

> The teacher of mathematics has a triple task: to fit the mathematical material to the state of development of the learners' mathematical schemas; to also fit the manner of presentation to the modes of thinking … of which the learners are capable; and finally to increase gradually the learners' analytic abilities to the stage at which they no longer depend on their teacher to predigest the material for them.

Investigation and Discovery Redux

In the investigation and discovery lesson that evolves out of the chicken problem, we see all these ideas at work. The class goal is to 'discover' how the known idea of direct variation can be extended to the more complex situation of joint variation. This type of lesson enables these learning considerations:

- Investigation and discovery is how mathematics is actually done—on half-slips of paper, in spurts of association and sudden, partial insights—not all at once, but in pieces until various loosely-connected ideas emerge into the full solution.
- Investigation and discovery legitimizes a transitory type of thinking: how to interact with the problem information, generating additional information, and the use of self-questioning/self-teaching that often makes accessible more finished insights and reasoning products—*after* the working solution is achieved.
- Unlike the term "problem solving," investigation implies that you must take **actions** when you don't know how to proceed, to generate new information not originally given. You explore the situation to discern how it operates, comparing your observations to similar ideas which you already know. *Discovery* occurs with **sudden illumination** when you discern how to modify your existing schema to meet the requirements of the new situation.
- The teacher's questions model how you approach mathematics, whether through learning new material (this lesson) or solving a problem you have never faced before. She takes into account the limitations of short-term memory, by separating the discussion about the process of thinking through the problem from its solution steps.
- She adheres to several goals, one at a time: sense-making, collaborative learning, connecting ideas, generalizing the process, how a professional (her) thinks through mathematics, student reading of the text, individual feedback the next day to discern where students' thinking is.

Investigation and discovery learning checks off several boxes with respect to what good mathematics instruction aims to accomplish. It is illustrative of what many good teachers do individually, on occasion, to bring a new spark to learning and to 'chip away' at the independence suggested by Skemp's third aim. It not only addresses the acquisition of mathematical knowledge, but also *how to learn, how to read technical materials, and how to navigate one's way around the subject of mathematics.*

But it takes time—out of the learning time available, and out of the teacher's schedule to develop such lessons. Its cumulative effect is too often restricted to the efforts of one teacher and one course, rather than a goal consistently addressed and organizationally delivered across the spectrum of 7-12. Which brings us to the purpose of this article, and its two linked articles to follow. At the end of his long and varied career, this author wishes to pose the first of three recommendations that he believes could drive long-term systemic improvement of secondary mathematics—if applied at the program level.

> **Recommendation 1** Give increased emphasis to investigation and discovery across grades 7-12 through the enacted curriculum, recurring learning opportunities, teaching methods, and materials.

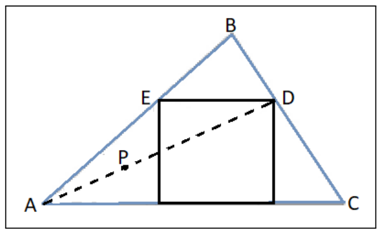
Linked Articles

Our strategy in this first article has been to identify a learning approach for leveraging change, systematically and systemically across 7-12 mathematics education. We will pick up the thread of this dialogue in the next article "Mathematics and Thinking" by building a case for change in teacher professional enrichment, while extending our discussion about the benefits of *Investigation and Discovery*. In our third article, "A Laboratory for Secondary Math" we focus our dialogue toward a place in the curriculum for investigation and discovery learning to occur and examples for consideration for how associated activities might be developed by commercial companies as a program of lab activities to accompany textbooks.

Conclusion

Our goal is that all students understand better how mathematics is done (independently) outside the classroom as well as in, including where mathematics comes from. Our vision is that better students exiting twelfth grade should be able to look at Polya's solution and appreciate how such solutions arise, *You know . . . he probably didn't do it that way the first time. It is more likely that he did this. He first drew a sketch of the required square in its approximate position (always a good first step). Selecting a point D, he dropped a perpendicular to AC, a parallel to AC and then another perpendicular from E to complete the square. The problem then became: how to locate the point D?*

Figure 2: *Student sketch of Pólya's idea.*



Looking at the diagram and not knowing what to do, he decides to draw in the segment AD – not for any specific reason, except that it is always a good thing to have more triangles from which to look for relationships. Then, he probably stared at the diagram for A LONG TIME before he thought about that point D moving down that dashed line segment to point P, saying to himself . . . hmm, it almost looks like a square could be constructed from P in the same way it was from D. I wonder . . . could this be true?

He checks and the working solution is born. Later, Pólya explains how the same result could be accomplished in the manner depicted, making for a more elegant solution.

A Historical Tidbit

The chicken problem is a familiar friend to veteran teachers across the span of many decades. Such puzzle-type problems have a compelling appeal to students and have the advantage to teachers of problem situations that involve small numbers for easy manipulation to discern the underlying mathematical ideas.

The Wall Construction problem, concluding the discovery lesson, is of historical interest not only because it presents a joint variation, similar to the chicken problem – but that it comes from an 1855 elementary mathematics book formerly used in this country. Within that historical tome, problems that involve joint variation are not noted as such, nor solved in the way depicted in our discovery lesson.

Rather, they are solved as direct variation problems, considering one independent variable at a time—which must be solved in succession to produce the requisite answer. [A simple direct variation is shown to be solved by a proportion of two like-type ratios for which three parts of the proportion are known and the fourth can be determined by a method called 'The Rule of Three.']

$$a : b :: c : d \quad \text{or} \quad \frac{b \times c}{a} = d \quad \text{(rule of three)}$$

Using this process on the chicken problem, we focus first on the effect of increasing the number of chickens. Second, a similar proportion is used to account for the effect of increasing the number of days, while now incorporating the intermediate number of eggs produced from the first proportion. (We use denominate numbers to identify the like-type ratios.)

$$1\frac{1}{2} \text{ chickens} : 12 \text{ chickens} :: 1\frac{1}{2} \text{ eggs} : \text{1st answer} \quad \frac{12 \times 1.5}{1.5} = 12 \text{ 1st answer (12 eggs)}$$
$$1\frac{1}{2} \text{ days} : 12 \text{ days} :: \text{ 1st answer} : \text{true answer} \quad \frac{12 \times 12}{1.5} = 96 \text{ true answer (96 eggs)}$$

The book next shows this process as one computation and calls it the "Rule of Six" noting that the procedure can be adapted to a "Rule of Nine" and so on.

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