A Laboratory for Secondary Math*

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Abstract: *Students should have opportunities to learn in representative formats for which they will eventually have to apply their knowledge. That is the premise of various curricular innovations over the past 30 years, including authentic learning, problem solving, rich problems/tasks, collaborative learning, modeling, numerical/quantitative literacy, technology, cooperative projects, or STEM-related learning. It also includes tasks related to traditional learning presented in an investigate and discovery format. This article argues for "a place" in the curriculum wherein students and teachers assume a virtual laboratory approach, at times, for learning across the secondary level as an organizational feature, and as an incentive for publishers to provide the activities and tools required to support application of learning for each secondary mathematics subject.*

Keywords: *Laboratory learning approach, Secondary mathematics education, Non-routine problem-solving*

Introduction

> Nobody gets a job factoring polynomials. Nor does an employer hire someone just because they know certain geometry theorems. What they do pay for are people who can think their way through problems successfully, many of which have never been specifically studied in a classroom by the prospective employee. They want workers who can reason and read technical materials; employees who may need to be retrained a number of times over their career; people who can work independently at times and as part of a team at others; and workers who possess traits like perseverance, resilience, curiosity, and adaptive expertise.

As a general rule, students learn what they have the opportunity to learn and practice. The contrapositive is also true for most students. What traditional lessons do well is enable students to learn new mathematics content to a beginning (sometimes intermediate) level of proficiency under the tutelage of a teacher. What those lessons do not do so well is to equip students to apply independently what they have learned to problem settings and situations that have not been studied.

We start with this premise: **Students should have opportunities to learn in representative formats for which they will eventually have to apply their knowledge.** Moreover, these opportunities should not be postponed to some 'future time' when students have met some designated proficiency or maturity of thought. They should be an integral part of each course at the secondary level. This leads to our third recommendation for the three linked articles by this author.

> **Recommendation 3**: Secondary mathematics courses should incorporate a virtual lab into the instructional framework to provide 'time and a place' for students to experience mathematics as the process of applying the knowledge they possess in new situations in which they play the principal role.

Editor's Note**: This article is the third of three linked articles by this author, published by the OJSM over the most recent three issues. The first article, ***Investigation and Discovery, suggested using this framework consistently across courses to emphasize the mindset one employs in doing mathematics, whether learning new content or applying what one already knows to situations yet to be encountered. ***Mathematics and Thinking*** discussed the implications of beliefs and habits of thinking to learning, as well as how the mind is trained by mathematics to approach thinking challenges. In this third article, the author expands these notions by identifying 'a place' in the curriculum where students and the teacher accept different roles in the learning process.

Why is a Laboratory Needed?

Some types of lessons have more potential for student learning than others. These lessons engage students to a higher degree in their own learning. They provide insights into the nature of the subject as a discipline and how it is practiced. They are memorable in students' minds and spark conversations transcending the classroom as students collaborate in the learning activity. They add a dimension to learning that their more pedestrian cousin lessons do not.

If these lessons were encountered in the discipline of science, they would be called *laboratory lessons*. This is what science has to say about the unifying principle of inquiry, which students practice in the laboratory:

> [Scientific] inquiry means that students are handling science; they are manipulating it, working it into new shapes and formats, integrating it into every corner of their world, and playing with it in unknown ways. Inquiry implies that students are in control of an important part of their own learning where they can manipulate ideas to increase understanding. As students learn to think through the designs and developments of their own inquiry, they also develop a sense of self-responsibility that transcends all subject areas. (Just Science Now, 2020).

By substituting mathematics for science and investigation and discovery for scientific inquiry, we begin to imagine a different type of mathematics learning delivery in a virtual setting—where the rules of engagement are different. Where students and teachers, on occasion, accept different roles from the traditional classroom experience and act their parts accordingly. The purpose of laboratory lessons is to engage students in thinking more deeply and for longer periods of time about the mathematics under consideration.

Consider for a moment a specific course and imagine answers to these questions: *What would a laboratory lesson look like? What aspects of mathematics would the lab address? What would be the relationship of the lab lesson to mainstream lessons taught during the other days? How would the actions of teacher and students change? What thinking might be better taught and learned in an investigative way? How would this new dimension address unmet needs from the student perspective? The future employer perspective? The teacher and other stakeholders' perspectives?*

An Idea Whose Time Has Come

For more than three decades, individual teachers have sought ways to supplement the customary lesson format of presentation and practice with applications of the mathematics being learned in ways that illustrate how practitioners utilize mathematics knowledge to accomplish meaningful tasks. The questions above are considerations that various curricular innovations attempt to answer for: authentic learning, problem solving, rich problems/tasks, collaborative learning, modeling, numerical/quantitative literacy, technology, cooperative projects, or STEM-related learning, to name a few. Each was born of the notion that formalized education is a temporary learning environment to prepare us for a world beyond schooling. Following are three selected sources that support the kind of authentic learning that a laboratory suggests.

Work environments contrasted to school

University of Pittsburgh researcher Lauren Resnick (1987) identified three ways in which the way we learn in schools differs from how education is applied in a work environment. School environments emphasize individual seatwork, whereas adults must work together collaboratively in many environments, sharing their expertise, insights, assigned tasks, and decision-making. Practical work environments rely heavily upon tools and other technology appropriate to a limited number of specialized tasks. Schools focus primarily on mental tasks that are constantly changing daily. Abstract reasoning, supplemented by the learning of procedural knowledge, characterizes most classrooms.

Everyday environments incorporate contextualized reasoning about specific tasks in well-defined applications. Workers often apply quite different procedures than those learned in schools.

The 21st century has upped the ante on worker adaptability. Automation rapidly displaces specific skills that can be acquired through a one-time schooling experience. Jobs that require humans increasingly demand complex reasoning, problem-solving, teamwork, and technology-rich expertise. But even those jobs will evolve as the rapid pace of change continues. More education and training will ensue. One of the most important skills that students can acquire is the ability to learn how to learn.

Rich problems and collaboration

Learning formats that bridge the classroom and work worlds are described in various terms. Audrey Rule (2006) of the State University of New York (SUNY) at Oswego identifies four themes repeatedly found in authentic learning across disciplines:

1. An activity that involves real-world problems and that mimics the work of professionals.
2. Use of open-ended inquiry, thinking skills, and metacognition.
3. Students engage in discourse and social learning in a community of learners.
4. Students direct their learning in project work.

Steve Hewson and Jennifer Piggott of NRICH (University of Cambridge) describe rich mathematical tasks thusly: Rich tasks open up mathematics. They transform the subject from a collection of memorized procedures and facts into a living, connected whole, allowing the learner to 'get inside' the mathematics. The resulting learning process is far more interesting, engaging, and powerful; it is also far more likely to lead to a lasting assimilation of the material for use in further mathematical study and the wider context of applications (NRICH, n.d.). A sampling of rich-task features is that they:

- are set in contexts that draw the learner into the mathematics because the starting point or the mathematics that emerges is intriguing
- are accessible and offer opportunities for initial success, challenging learners to think
- offer different levels of challenge to match the learner's level with the potential to extend those who need and demand more (low threshold - high ceiling tasks)
- allow for different methods and different responses.

Authentic assessments

The increasing trend across state, national, and international assessments is to measure not only what students know but also what they can do. The *Programme for International Student Assessment* (PISA) is a worldwide study by the Organisation for Economic Co-operation and Development (OECD) to evaluate educational systems by measuring 15-year-old school pupils' scholastic performance in mathematics, science, and reading. This is an excerpt from the definition of mathematical literacy used in the 2015 draft mathematics framework (p.5):

> As the basis for an international assessment of 15-year-old students, it is reasonable to ask: "What is important for citizens to know and be able to do in situations that involve mathematics?" More specifically, what does competency in mathematics mean for a 15-year-old? . . . the construct of mathematical literacy . . . is used in this report to denote the capacity of individuals to formulate, employ, and interpret mathematics in a variety of contexts . . . it is intended to describe the capacities of individuals to reason mathematically and use mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena (PISA, 2012).

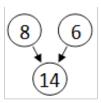
What Might a Mathematics Lab Activity Look Like?

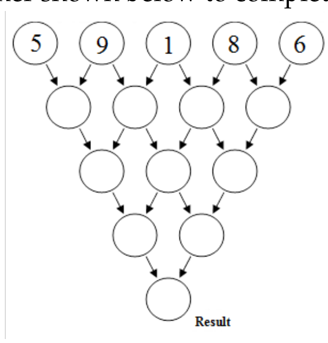
There are many possibilities for learning opportunities that cast students into the active roles of investigation and discovery. The key starting point is that the task showcases something new and worthwhile for the student to know. Additional considerations are (1) situations that capture students' imaginations in new ways to think, interact with one another, and appreciate how the ideas of mathematics can be wielded effectively; (2) activities that permit the learner to entertain them at a variety of entry levels; and (3) written responses that ask the student to reflect on what he or she has learned. We offer a couple of illustrations and suggest a few others.

Sets of Non-Routine Problems

Rudd Crawford of *Stella's Stunners* fame has had great success in presenting students with weekly sets of 6-8 problems that extend classroom ideas in ways that are new to students (Crawford, 2021). One or two problems require deeper reflection of the following type. Problems are carefully selected or constructed to require investigation to make headway while offering the students rich discoveries about mathematics, their thinking, and the experience of marshaling one's intellectual resources over a different time frame than an exercise.

1. (a) Add in the manner shown below to complete the diagram on the right:





   (b) Does the final result at the bottom change if the order of the numbers in the top row is different?

   (c) Find five different whole numbers 1-9, which, when entered in the first row, will produce a result of 100.

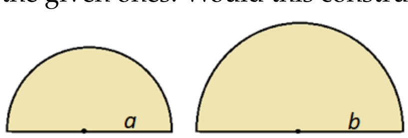
   (d) Can you find a way to predict the final result by knowing the beginning numbers and their order?

2. Observe that $50^2 - 50 = 49^2 + 49$ For what other pairs of numbers is this form of relationship true? Why must it be true?

3. Triangle ABC has sides with lengths 4 cm, 7 cm, and 9 cm. Construct triangle ABC and then find the lengths of its three altitudes. (a) State the relationship among the lengths of the altitudes (h_4, h_7, and h_9) by using inequalities; (b) Provide an argument for why this must always be the case; (c) State your findings as a theorem for any triangle with sides such that $a < b < c$ and having altitudes h_a, h_b, and h_c.

4. For any whole number $n \geq 2$, describe a general means to write its unit fraction $\frac{1}{n}$ as the sum of two other, different unit fractions. That is, $\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$ where x and y are whole number expressions involving n. Part II: Use this theorem to find sums of different unit fractions for these non-unit fractions: (a) $\frac{4}{5}$, (b) $\frac{2}{7}$, (c) $\frac{10}{11}$.

5. Given semi-circles with respective radii a and b, describe how to construct a third semi-circle with radius c such that the area of the constructed semi-circle is equal to the sum of the areas of the given ones. Would this construction method work for other figures? Explain.



Each problem is written in a notebook (whether a successful solution has been found or not). This author also adds the requirement of a **Take Away**. That statement concludes the writeup with: *What were your feelings (or thoughts) while engaging the problem? What has the problem taught you that you didn't know before? (About mathematics? About thinking? About yourself?) What has the investigation made you wonder about, beyond what is asked?*

For example, for Problem 1 in the previous section, we would hope that different students might note some of the following. That a formula permits one to search more efficiently for the correct combination of numbers in the appropriate positions. That some positioning of numbers (1st and 5th; 2nd and 4th) are isomorphic. That coefficients in the formula (and the process) remind one of Pascal's triangle. That some students might be motivated to ask, *What is the minimum (35) and maximum (125) sums that might be obtained?*

Project-Based Learning

In project-based and peer learning, students must collaboratively conduct experiments to generate their own data. Then analyze and reconcile empirical results with theoretical expectations. They are faced with new ways to think, interact with one another, and appreciate how the ideas of mathematics are connected.

Cracker Box (suitable for a geometry class)

Objective: To study scaling factors between similar area figures and similar volumes

Students are shown an ordinary cracker box and given the scenario that the boss wants to produce a new box that holds double the volume of their customary box. The new box should be similar in proportions to the old box. Student 'workers' are to determine the dimensions for the new box and to construct a model of it from the poster sheet provided.

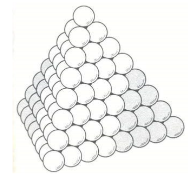
In the directed part of the activity, students are given a drawing of the box manufactured by the cracker company on isometric graph paper. They are asked what happens when a scaling factor is applied to each measurement of the box—namely, how does that change the volume? The teacher suggests doubling each dimension on the isometric graph paper and determining how it affects the volume. The class works through the derivation shown in the diagram to see why this happens, concluding that they must find a scaling factor k such that $k^3 = 2$ and $1 < k < 2$. They then set about the task in pairs to disassemble a box they have brought from home into its net, taking measurements, finding the correct k-value, laying a new net for the box with the doubled volume, and reassembling. The work flows into the homework over the next two days. The activity concludes with pairs demonstrating with packing peanuts the volume comparisons of their original box to the constructed box.

$$\begin{array}{ll} \text{suppose} & V_o = l_o w_o h_o \\ \text{doubling} & V_n = 2l_o \cdot 2w_o \cdot 2h_o \\ & V_n = 2^3 \cdot l_o w_o h_o \\ & V_n = 8\ l_o w_o h_o \\ \text{or} & V_n = 8\ V_o \\ & \\ \text{we want} & V_n = 2\ V_o \end{array}$$

As a capstone, each student is presented with an award certificate (***Certificate of Awesomeness***), which their mothers decide must be reduced in size to be one-half the original area. Students' concluding task is to determine what setting to use as the reduction factor on a copy machine to accomplish this task and what the reduction factor applies to—area or length and width measures.

Cannonballs and Finite Differences (suitable for a second semester Algebra II or Adv. Math class)

Objective: To learn how the method of finite differences can be applied to determine a polynomial function's equation from suitable data points.



Students are given teacher-prepared sheets to read and discover the method of finite differences (not explained here, see *Exhibit 1: Cannonballs, Finite Differences, and Sequences* at the following link) for finding a polynomial function to describe the data integral to a practical situation—in this case, stacks of cannonballs with either a triangular or square base. This is an excerpt of the prompt:

There are occasions we would like to generate a formula to fit a given number of data points that seem to fit a regular pattern of occurrence. One such method for doing so can be accomplished if that pattern can be described by a polynomial function, for example, $f(x) = ax^3 + bx^2 + cx + d$, and provided that we have data pairs $(x, f(x))$ of successive integral values of the independent variable x.

Working through examples, students discern the general correspondence of the degree of the polynomial function and the number of difference columns required to produce a constant difference. Eventually, they apply what they have learned to the cannonball figure. In this instance, they generate this formula $S_n = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$ which can be factored as $S_n = \frac{1}{6}n(n+1)(n+2)$.

Finally, pairs apply what they have learned to find formulas for (a) a stack of cannonballs with a square base; and (b) predicting the number of cubes to construct a pyramid of cubes stacked into a corner. Comments: This type of laboratory format usually starts in the lab period and is completed within a day or two outside of class. It is often self-evaluating in that when the particular task is completed correctly, it demonstrates its own correctness. Students usually work with at least one partner, learning new mathematics or ways to apply previously learned ideas in a context. Learning outcomes, often given only lip service, like reading mathematics, how to learn, how to collaborate with others on a task come into play. The teacher is usually a facilitator in the process, sometimes launching the project and on hand to answer questions and get students unstuck.

Other Formats Useful for a Laboratory

(1) Rich Problems and Tasks; (2) Released Assessment Item Practice; (3) Selected Units from Innovative Projects; (4) Discovery Lessons/Discussions of Thinking.

Conclusion

Why a Laboratory Designation? For students, it signals a different format for learning with different expectations. The accepted rules for an activity under lab consideration are: (1) it may not be immediately evident how to proceed; it must be investigated; (2) the emphasis is on thinking your way through the situation; (3) you may be working collaboratively with one or more partners, for a sustained period of time, extending beyond the classroom; (4) a write-up of your investigation will be required whether successful or not; (5) you will be asked to reflect on the learning task as to how its investigation has changed your thinking.

For the teacher, it means making a commitment toward a learning format where students assume the primary roles. It may not be easy to implement at first, as most students tend to conform to the path of least resistance—expecting the teacher to do the 'heavy lifting.' The experience must start slowly and err on the easy side, until students accept the different expectations. Evaluation and motivations must be rethought, with periodic feedback sought from students about their reactions and feelings. Time to fit the laboratory experience into the curriculum must be managed so that new content learning does not suffer.

Programmatically, it is a step toward tracking learning outcomes of a broader and more enduring nature. It suggests discussions across grades— *where do we address helping students learn how to read mathematics, learn independently, how to study mathematics, how to work collaboratively? How do our individual philosophies of teaching impact students as they progress from grade to grade or course to course?*

How does learning translate beyond assessment performance to job preparation and success at the next level? It makes it a pragmatic step to ask publishers: 'What tools beyond the textbook do you offer respecting these broader outcomes? for laboratory learning?'

Most importantly, the type of learning discussed across our three linked articles is the raison d'être for the study of mathematics at any level, at any time. Beyond a body of content, mathematics is a way of thinking. A kind of know-how to address questions of one's own selection and importance. It consists in developing habits of thinking that can open up new windows of possibility and discovery, providing the wielder with a sense of empowerment over what has been learned. A growing understanding about how to wield one's mind, a sense of wonder about what mathematics offers, and the elation of discovery in navigating through it under one's own direction. This is not something to be withheld until some future time; it must be celebrated as part of every course.

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Additional Resources

Meiring, S.P. (unpublished manuscript) *ThinkAbout It: A Resource Guide for Secondary Mαth Teachers and Students*.

Ohio Resource Center. This website is a source for rich problems/projects: https://web.archive.org/web/20121017083539/http://www.ohiorc.org/for/math/problem_corner/default.aspx

Search tool for 2000 released NAEP items: https://nces.ed.gov/nationsreportcard/pdf/naep_nqt_final_web.pdf

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