Visualizing an Alternating Series

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Abstract: *The author explores the convergence of an alternating series using a visual approach. Ideas for engaging students are provided along with a formal convergence proof.*

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Introduction

We begin with an alternating geometric series:

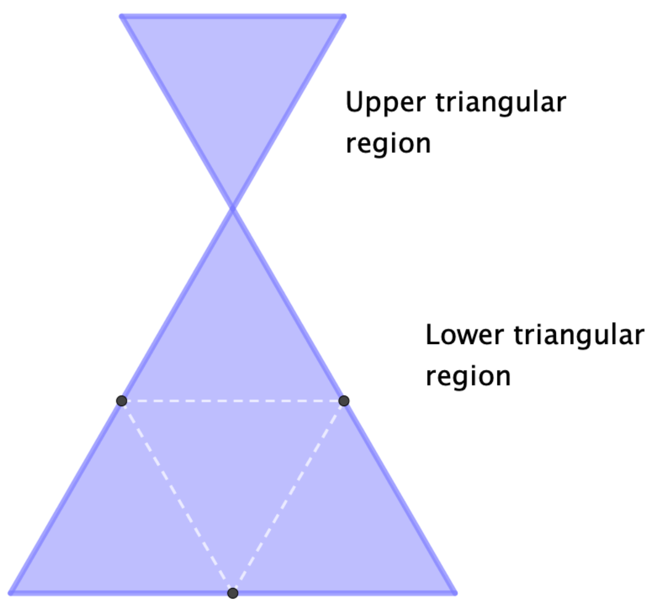
$$\sum_{n=1}^{\infty}\left(\frac{-1}{4}\right)^{n-1} = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \frac{1}{256} - \dots$$

In addition to exploring possible convergence of the series numerically, we use a visual approach to better address the varied learning needs of our students.

Representing the First Term Visually

With this particular series, we represent the first term (i.e., 1) with the shape shown in Figure 1.

Figure 1: *Visual representation of the first term of the series.*

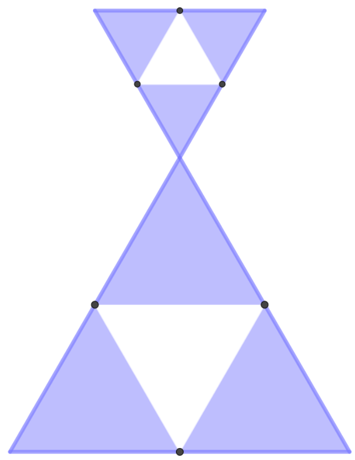


We've added dotted lines to the figure to emphasize that the area of the upper triangular region is one-fifth of the area of the entire shape. Note, too, that the "upper" and "lower" triangles are similar, with a ratio of 1:2 from upper to lower.

Representing the Second Term Visually

Considering terms of the sequence as areas, we represent $S_2 = 1 - \frac{1}{4}$ visually with the removal of medial triangles from the "upper" and "lower" parts. This is illustrated in Figure 2.

Figure 2: *Visual representation of the sum of the first two elements of the sequence (i.e., S_2).*



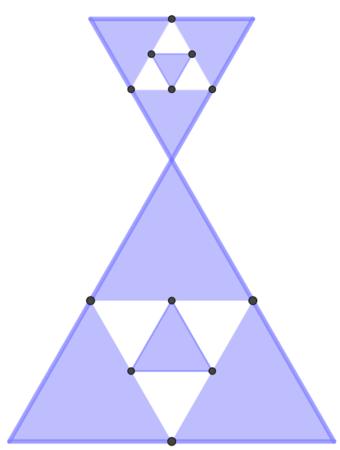
Since the first term has an area of 1 and is comprised of 5 congruent triangles, the area of the "upper" triangle is $\frac{1}{5}$ while the area of the "lower" triangle is $\frac{4}{5}$. The removal of the medial triangles reduces the area of each region by one-fourth. Thus the area of the second term is given by

$$\begin{aligned} S_2 &= 1 - \frac{1}{4} \cdot \frac{1}{5} - \frac{1}{4} \cdot \frac{4}{5} \\ &= 1 - \frac{1}{4} \end{aligned}$$

Representing the Third Term Visually

Since we're working with an alternating sequence, the next term will be added rather than subtracted. This time, we *add* medial triangles inside the "holes" that we created in our previous step as illustrated in Figure 3.

Figure 3: *Visual representation of the sum of the first three elements of the sequence (i.e., S_3).*



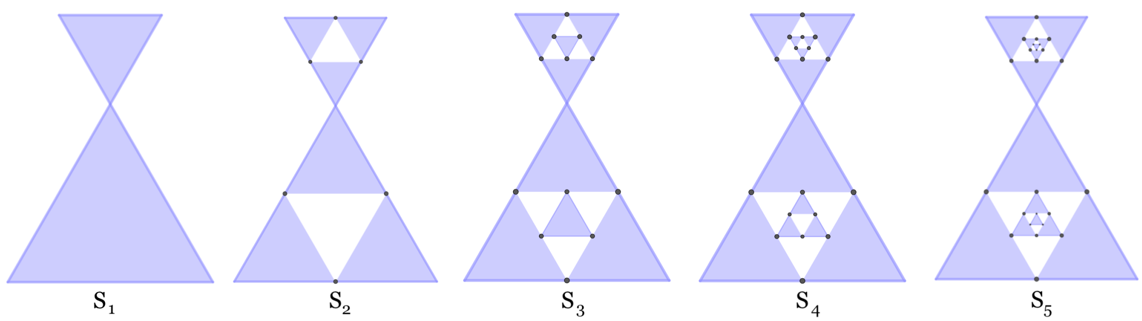
Note that we've covered precisely one-fourth of the "upper" and "lower" holes, yielding the following area for S_3:

$$\begin{aligned} S_3 &= S_2 + \frac{1}{4} \cdot top\ \ hole + \frac{1}{4} \cdot bottom\ \ hole \\ &= S_2 + \frac{1}{4} \cdot \left(\frac{1}{4} \cdot \frac{1}{5}\right) + \frac{1}{4} \cdot \left(\frac{1}{4} \cdot \frac{4}{5}\right) \\ &= S_2 + \frac{1}{16} \\ &= 1 - \frac{1}{4} + \frac{1}{16} \end{aligned}$$

Continuing the Process

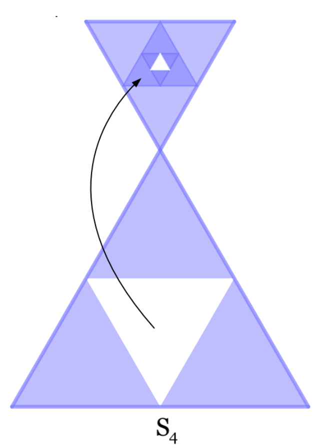
As we continue the process of adding and subtracting area, fractal-like designs begin to emerge. Figure 4 illustrates the first 5 partial sums of the series (i.e., S_1 through S_5).

Figure 4: *Visual representation of the first 5 partial sums of the series (i.e., S_1 through S_5).*



The images in Figure 4 clearly illustrate that the areas of the sums are bounded. Furthermore, moving pieces from the "lower" hole to the "upper hole" suggests $\lim_{n\to\infty} S_n = \frac{4}{5}$. Consider, for instance, the rearrangement of S_4 illustrated in Figure 5.

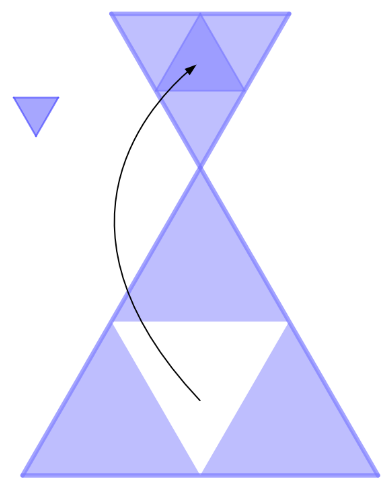
Figure 5: *When pieces are moved from the "lower" to "upper" region, we see that $S4$ is slightly less than $\frac{4}{5}$*



When k is even, the last term of S_k is negative, making $S_k < \frac{4}{5}$. On the other hand, when k is odd, the last term of S_k is a positive value, making $S_k > \frac{4}{5}$.

In Figure 6, we use a visual argument to illustrate this for S_3. When we move pieces from the "lower" to "upper" region, we completely fill the "upper" hole—and have a small triangular piece left over.

Figure 6: *When pieces are moved from the "lower" to "upper" region, we see that $S3$ is slightly larger than $\frac{4}{5}$*



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The oscillation around the limit of $\frac{4}{5}$ is suggested by the first few partial sum calculations:

$$S_1 = 1 > \frac{4}{5}$$
$$S_2 = 1 - \frac{1}{4} = \frac{3}{4} < \frac{4}{5}$$
$$S_3 = 1 - \frac{1}{4} + \frac{1}{16} = \frac{13}{16} > \frac{4}{5}$$
$$S_4 = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} = \frac{51}{64} < \frac{4}{5}$$

Convergence Proof

Note that our series is geometric. The first term is $a = 1$, while each subsequent term is found by multiplying the previous term by the common ratio $r = -\frac{1}{4}$.

A well-known formula for calculating a geometric series with $r < 1$ is given by $\frac{a}{1-r}$ (Wikipedia, 2022). Using this formula with our series yields the following.

$$\begin{aligned} S_\infty &= 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \frac{1}{256} - \dots \\ &= \frac{1}{1 + \frac{1}{4}} \\ &= \frac{1}{\frac{5}{4}} \\ &= \frac{4}{5} \end{aligned}$$

Exploration Ideas for Students

The ideas presented within this paper are useful for exploring a wide variety of series. For instance, suppose we modified the geometric interpretation of the initial term to consist of similar hexagons, as shown in Figure 7.

Figure 7: *Replacing triangles with hexagons in our original series.*

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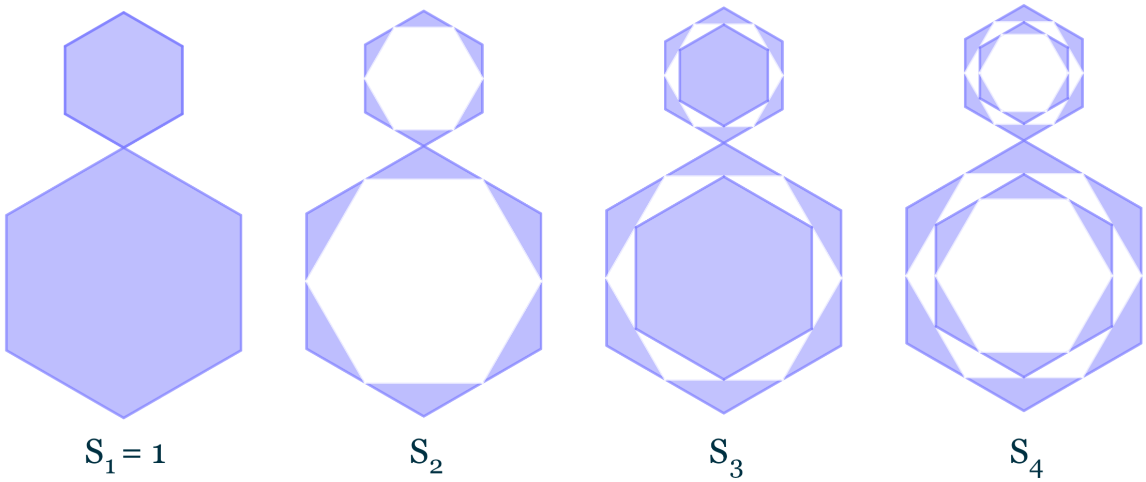

Note that the visualization in Figure 7 was generated by adding (or removing) medial hexagons at each step. Teachers can provide students with such a visual and ask them to generate a corresponding numerical series. For more of a challenge, teachers could provide their students with a numerical sequence and ask them to construct a visual representations. For instance, how would *you* represent the following series graphically?

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \dots$$

References

Wikipedia (2022, October 14). *Geometric series.* Available on-line at https://en.wikipedia.org/wiki/Geometric_series

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