The Sum of Even Powers

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Abstract: *The authors explore a proof involving the sum of even powers using a visual approach. Ideas for engaging students are provided along with a formal proof.*

Keywords: *Proof and argumentation, visualization, sums of even powers*

Introduction

It was known to Pythagoreans that $\sum_{i=1}^{n} i$ is given by the corresponding triangular number. The closed form of $\sum_{i=1}^{n} i^2$ was discovered by Archimedes, and of $\sum_{i=1}^{n} i^3$ by Aryabhatiya, while the earliest proof for it is due to Abu Bakr al-Karaji. Abu Ali al-Hasanibn al-Haytham was the first to find the closed form for$\sum_{i=1}^{n} i^4$.

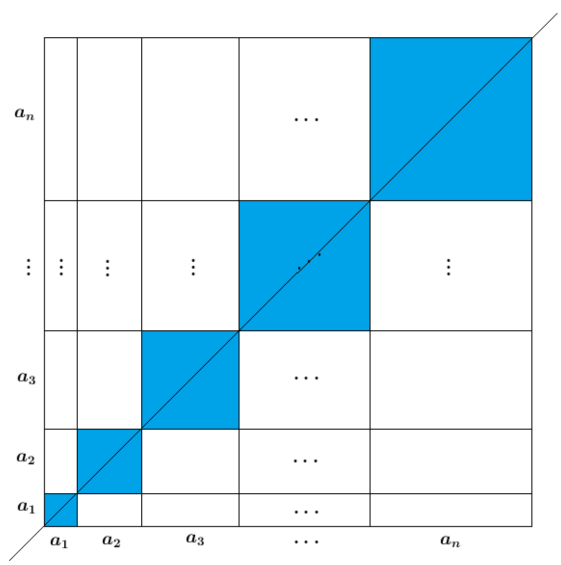
The aim of this article is twofold. First, we provide a visual proof of $\left(\sum_{i=1}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j$, where a_i $(i = 1, 2, \ldots, n)$ be n positive real numbers. Later, we will show a connection between $\sum\limits_{i=1}^{n} i^{2k}$ and $\sum\limits_{i=1}^{n} i^k$.

Results

Let a_i $(i = 1, 2, \ldots, n)$ be n positive real numbers. Then

$$(\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i=2}^{n} (a_i \cdot \sum_{j=1}^{i-1} a_j).$$

Proof.



Remark 1: Note that the each side of the bigger square is $(a_1 + a_2 + \ldots + a_n)$. The above figure demonstrates that area of the bigger square is the sum of n rectangles (n-columns). Thus

$$(a_1 + a_2 + \ldots + a_n)^2 = \sum_{i=1}^{n} a_i \cdot (a_1 + a_2 + \ldots + a_n).$$

Moreover, the figure is symmetric about the secondary diagonal (see the figure). Thus the area of the bigger square equals the sum of the areas of the smaller squares shaded with blue color plus twice the sum of the areas of the $(n-1)$ columns below the secondary diagonal). In other words,

$$(a_1 + a_2 + \ldots + a_n)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i=2}^{n} a_i \cdot (a_1 + a_2 + \ldots + a_{i-1}).$$

Remark 2: Since the figure is symmetric about the secondary diagonal, thus the area of the ij-th $(i < j)$ the rectangle is same with the area of the ji-th $(i < j)$ rectangle. Thus if a_i $(i = 1, 2, \ldots, n)$ be n positive real numbers, then the above figure demonstrates that

$$\Big(\sum_{i=1}^{n} a_i\Big)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j.$$

Remark 3: If we take $a_i = i^k$ for $i = 1, 2, \ldots, n$ (k is a natural number), then we get the following (Edgar, 2019).

$$\sum_{i=1}^{n} i^{2k} = \left(\sum_{i=1}^{n} i^k\right)^2 - 2\sum_{i=2}^{n}\left(i^k \cdot \sum_{j=1}^{i-1} j^k\right),$$

Note to Teachers

Students have seen these types of problems and figures at their early school level studies. Basically, when they learned the formula $(a+b)^2 = a^2 + 2ab + b^2$ early in their mathematical journey, many teachers explained this formula using a square with side $a + b$. This figure prescribes the more general formula for $(a_1 + a_2 + \ldots + a_n)^2$. (See, Remark 2). These are the visual exercises for them to visualize the formulas for $(a-b)^2$ and $(a \pm b)^3$.

On the other hand, this figure explains a connection between $1 + 2 + 3 + \ldots + n$ (total area of the figure) and $1^2 + 2^2 + 3^2 + \ldots + n^2$ (blue shaded region). Thus students make attempt to derive the formula for $1 + 2 + 3 + \ldots + n$ using similar type figures.

Also, it may be an amazing visual exercise for them to prove the AM-GM inequality ($\frac{a+b}{2} \geq \sqrt{ab}$ where $a,\ b \geq 0$) using similar kind of figures.

References

Edgar, T. (2019). Proof Without Words: Sums of Even and Odd Powers. *Mathematics Magazine, 92*(4), 300–301. https://doi.org/10.1080/0025570x.2019.1638215

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