Nurturing Structural Thinking through Teacher-Facilitated Problem Solving

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Abstract: *One essential goal of mathematics teaching is to develop the habit of mind and the ability to look for and recognize structures, to probe into and act upon structures, and to reason and justify in terms of general structures. Framed by the five practices for orchestrating productive mathematics discussions (Smith & Stein, 2011), this paper uses the Horse Rider problem as an example to illustrate how teachers can nurture student structural thinking through careful sequencing of mathematical tasks.*

Keywords: *Problem solving, rich tasks, structural thinking*

A rider is going from A to B on a horse, but first he wants to stop at the river to let the horse drink. A is 3 kilometers from the river, and B is 8 kilometers from the river. A and B are on the same side of the river and are 13 kilometers apart. What is the shortest path the rider can take?

The Horse Rider problem above is one version of Heron's Shortest Path problem, one of the first non-trivial optimization problems. The task was solved by Heron of Alexandria, who lived around 10–75 CE and is known to have written works on geometry and mechanics. Heron realized that if the river is a mirror, then the shortest distance connecting the mirrored image of point A and the destination B is given by a straight line. He concluded that its intersection with the river is the desired point.

Having students solve problems that interested early mathematicians is one approach to historically enriching the teaching and learning of mathematics. Meanwhile, it is important to understand that obtaining a solution to a problem does not have to be the end goal of problem-solving in mathematics classrooms. Rather, it can serve as a starting point for reflecting, extending, and connecting mathematical ideas. In other words, problem-solving can be used as a pedagogical means to support students in perceiving mathematics as a field of intricately related structures rather than a series of computations to be carried out. Recognizing that the development of students' structural thinking must begin long before they enter college, the Common Core State Standards for Mathematics (2010) includes looking for and making use of structure as one of the eight standards for mathematical practice. Framed by the five practices for orchestrating productive mathematics discussions (Smith & Stein, 2011), this paper uses the Horse Rider problem as an example to illustrate how teachers can nurture structural thinking through facilitating students to solve a sequence of carefully designed mathematical problems.

Selecting and Organizing Mathematical Problems

Like any other way of thinking, structural thinking is developmental in nature and evolves gradually through various mathematical activities. Although in many mathematics classrooms, students have substantial opportunities to solve mathematical problems, too often these tasks aren't organized in a way that promotes structural thinking (Bass, 2017). The process of seeking a solution to a mathematical problem often mobilizes particular mathematical ideas, which can be utilized to form a mathematical structure that might subsequently be used to solve a class of problems. Therefore, teachers can purposefully select and organize mathematical problems in a manner that orients students towards looking for and making use of structure.

An **anchoring** problem is a problem a teacher chooses to guide students to notice an intended mathematical structure that is generalizable across mathematical contexts. The Horse Rider problem is an example of an anchoring problem. Since an intended mathematical structure is introduced through solving a particular problem, students might only relate the mathematical model to problems that share a similar mathematical context with the anchoring problem. Therefore, it is important to provide students opportunities to work on extension problems that appear to come from totally unrelated topics in mathematics but can actually be solved using the same mathematical model in the anchoring problem. This type of problem can be labeled as a **horizontal extension** problem. Researchers have used "theory-building tasks" (Bass, 2017) and "nontrivial equivalent problems" (Usiskin, 1968) to describe this type of extension problem. These mathematically related problems demonstrate perhaps one of the most important qualities of mathematics, that is, the generalizability of one mathematical concept across contexts. Even when students perceive the generality of a mathematical structure, they might still not be able to reason with the general structure (Mason et al., 2009). Reasoning with a general structure is cognitively more demanding because it requires students to first abstract a general structure from concept images and then conceive it as a conceptual entity that one can act upon in thought. Therefore, students need to have the opportunity to work on **vertical extension problems**. They are problems that are created by varying the condition(s) of an original problem but can be solved by extending the reasoning used to solve the original problem.

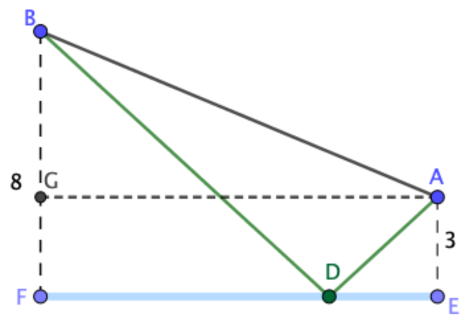
Anticipating Students' Responses and Monitoring Their Work to Draw Their Attention to a Mathematical Structure

Once an anchoring problem and its extension problems are selected and organized, it is important for teachers to anticipate how students might mathematically approach the anchoring problem. This involves considering how students might interpret the problem, different strategies they might use to tackle it, and how each strategy connects to the mathematical structure the teacher would like his or her students to learn. The anticipated strategies for solving a specific problem might come from different venues, including teacher's independent work on the problem, their collaborative work to solve the problem with colleagues, student responses on the same or a similar problem in the past, student responses documented in research, and so on. It is also important to consider new approaches and actions students might take because of access to digital technologies. When an anchoring problem is given to students, the teacher can monitor students' work when they work either individually or in small groups. This involves attending to students' mathematical thinking and solution strategies, helping students move forward in their problem-solving process, and drawing students' attention to mathematical structures that explain why their methods work.

The Horse Rider problem was implemented within a unit on geometric transformations in a high school geometry class, in which GeoGebra was frequently used as a teaching and learning tool. Students first worked in small groups in a pre-made GeoGebra file that modeled the problem situation. When students were working in small groups, the teacher circulated among different groups to monitor their progress by listening to group discussions, asking questions to assess or advance their thinking, and offering suggestions if students were stuck. The short excerpt below shows how the teacher guided the students to use the Pythagorean theorem to find the horizontal distance between points A and B when they were stuck. The Pythagorean theorem was later used by Group A to develop their method for solving the problem.

TEACHER : [*Referring to Figure 1*] The distance between A and B is 13 kilometers, A to the river is 3 kilometers, and B to the river is 8 kilometers. What's the vertical distance from B to A?
STUDENTS : Five kilometers because eight minus three is five.
TEACHER : Now we know BG is 5 and AB is 13. How can we find the horizontal distance between A and B?
JOE : Pythagorean theorem! So AG is 12 because 5, 12, and 13 form a Pythagorean triple. Now EF is also 12 because $\overline{AG}$ and $\overline{EF}$ are congruent.

Figure 1: *The horizontal distance between A and B.*



In another instance, after the students in Group B showed the teacher how they dragged D to find the shortest path, the teacher asked students what they noticed about the two right triangles when the path was the shortest. This question triggered the students to measure the angles in the two right triangles and to conjecture that the two right triangles were similar when the path was the shortest.

After extensive exploration in small groups, the teacher asked the groups to share their methods. Three approaches are presented here to show students' diverse responses to the problem.

Method by Group A

AMY : If DE is x, then DF is $12-x$. We used the Pythagorean theorem and found AD is $\sqrt{x^2+9}$ and BD is $\sqrt{(12-x)^2+64}$ (Figure 2a). So the total distance is AD plus BD. But we kinda got stuck there for a while. Then we decided to graph it and found the vertex of the parabola. So, the shortest distance is 16.28 kilometers and x is 3.27.

Method by Group B

AIDEN : We first measured AD and BD and added the two distances. We then moved D and found the shortest distance. (*Pointing to triangles ADE and BDF in Figure 2b*) We thought the two triangles looked similar when the distance was the shortest. (*Pointing to $\angle ADE$ and $\angle BDF$*) So, we measured these two angles. They were not exactly the same but very close. We thought it was caused by rounding because if I move D slightly this distance (*referring to the sum of AD and BD*) doesn't change. We also moved A and B and the two triangles were still similar. So we can find the location without moving D around. We can first find the ratio of AE and BF and then divide the length of the river into the same ratio.

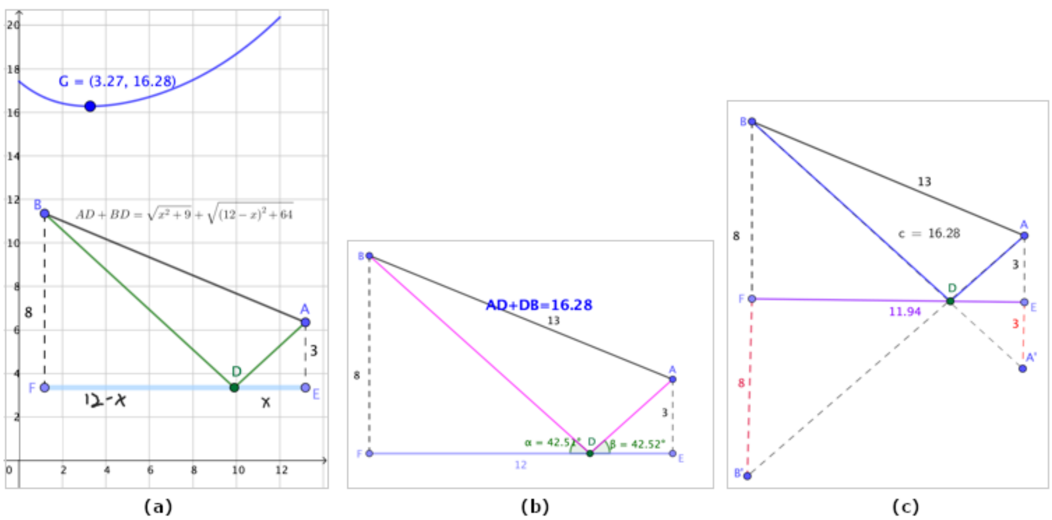
Method by Group C

RYAN : Reflect B over the river and then connect B and A. The point of intersection of $B'A$ and the river gives us the smallest distance from A to the river and then to B (*see Figure 2c*).

TEACHER : Very elegant! How did your group come up with this method?

RYAN : Well, since we are learning reflection, I was thinking about how to use it in this problem. So, I reflected A and B across the river. As we were moving D on the river, I noticed that when the distance was the shortest B', D, and A were colinear, and B, D, and A' were also colinear. So, to find D you can just connect B' and A. I guess you can also connect B and A'.

Figure 2: *Students' approaches to the Horse Rider Problem: (a) Method by Group A; (b) Method by Group B; (c) Method by Group C.*



Selecting, Sequencing, and Connecting Students' Responses to Reveal a General Mathematical Structure

Since a mathematical problem can often be solved with multiple strategies that make use of different mathematical ideas, different mathematical structures might be noticed and used by different students. By intentionally selecting, sequencing, and connecting students' responses, a teacher can bring the intended mathematical structure to the forefront and support students to connect their strategies with it. To nurture structural thinking, it is important to select responses that draw students' attention to the intended mathematical structure or particular elements of it. In the Horse Rider problem, after a few groups shared their methods, the teacher made use of the method presented by Group C and emphasized that the model was built on reflectional symmetry and the idea that the shortest path between two points is a straight line. This model instantiates the intended mathematical structure that the teacher wanted the class to learn. To connect Group A's method with this structure, the teacher drew students' attention to the fact that $AD = A'D = \sqrt{x^2+9}$ and, therefore, minimizing $AD + DB$ is equivalent to minimizing $A'D + DB$. Thus, $\sqrt{x^2+9} + \sqrt{(12-x)^2+64}$ is minimal when the value of x makes A', D, and B collinear. The teacher supported students to connect the method by Group B with the intended structure by mobilizing their knowledge of triangle congruency and similarity. The class reasoned that triangles ADE and BDF are similar when the path is the shortest because triangles $A'ED$ and BDF are similar by angle-angle similarity theorem, and triangles ADE and $A'DE$ are congruent because of reflectional symmetry.

Once a mathematical model is developed, it is important for students to discuss what elements are essential to the model and what can be varied without altering the underlying structure. This can help the students recognize the critical elements in the model and its generality. In the Horse Rider problem, by purposefully changing the location of A and B and the distance between A and B in GeoGebra, the students realized that the method would work as long as the two fixed points stay on the same side of the river. The students also noticed that the point on the river can be found by either connecting the mirrored image of A and B or connecting A and the mirrored image of B.

After the intended mathematical structure is abstracted from an anchoring problem, it is important to reveal its applicability to different contexts. The use of horizontal extension problems is a productive means to help students grasp the generality of the intended structure. Problem 1 is a horizontal extension problem of the Horse Rider problem. Although the problem asks students to find the smallest perimeter of a triangle, it is equivalent to finding the smallest sum of side lengths DF and EF since ED is fixed, which makes it mathematically identical to the Horse Rider problem (see the solution to Problem 1 at https://www.geogebra.org/m/c6yyf46v).

Problem 1

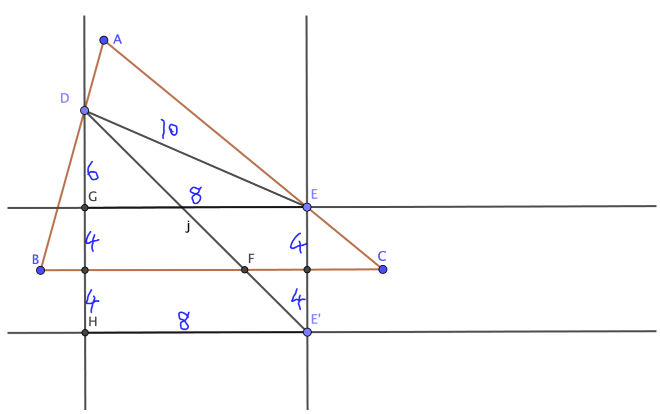
In an acute triangle ABC, D and E are fixed points on sides $\overline{AB}$ and $\overline{BC}$, respectively, and F is moving on side $\overline{AC}$. D is 10 cm from side $\overline{AC}$ and E is 4 cm from side $\overline{AC}$. The distance between D and E is 10 cm. What is the minimum perimeter of triangle ABC?

Problem 1 and other horizontal extension problems of the Horse Riders problem (see them here https://www.geogebra.org/m/eevftxgv) were first given to students as homework and then discussed in class. The following excerpt of class discussion on Problem 1 shows that the class had grasped the generality of the structure introduced in the Horse Rider problem.

TEACHER : Can any of you tell me how did you get $10 + 2\sqrt{65}$?

ALI : I just did the same thing (a few students nodded their heads). I reflected E over AC and connected E' and D (*the teacher started to replicate what Ali said in GeoGebra*). Then I used the Pythagorean theorem to find DE'. Actually, you have to use it twice, once to get the distance from E to that vertical line (*pointing to line $\overline{DH}$ in Figure 3*) and once to get DE'. So EG is 8 because DG is 6 and ED is 10. $E'H$ is also 8 because EG equals $E'H$. So, DE' squared is 8^2 plus 16^2, that is $2\sqrt{65}$ after I took the square root and simplified it. And then I added 10 to get the perimeter.

Figure 3: *Ali's solution to Problem 1.*

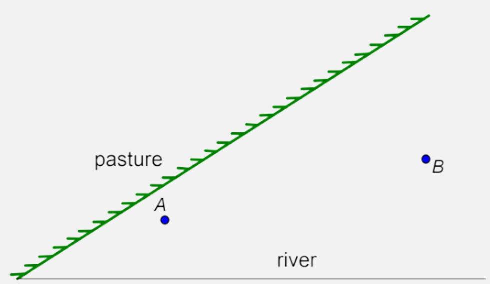


Reinforcing Students' Structural Thinking by Providing Them Opportunity to Reason with a General Structure

Even when students recognize the generality of a mathematical structure, they might still not conceive the general structure as a conceptual entity that one can reason within thought (Mason et al., 2009). The use of vertical extension problems provides a productive means to reinforce students' structural thinking because students are demanded reason with a general structure to solve these problems efficiently. The what-if strategy is useful for creating vertical extension problems that might be solved by generalizing the mathematical reasoning in the original problem. The notion of dimension of possible variation is helpful for enacting the what-if strategy (Mason 2003). Dimensions of possible variation are different aspects of a problem that are perceived as changeable. By intentionally varying a dimension or releasing a constraint in a problem, new problem(s) can be created. For instance, in the Horse Rider problem, adding a new stop creates Problem 2.

Problem 2

A rider is going from A to B on a horse, but first he wants to stop at the pasture to let the horse graze and then at the river to let the horse drink. Describe the shortest path he can follow.



Problems 2 and another vertical extension problem were given to the students after they solved a few horizontal extension problems (see solutions to the two problems at https://www.geogebra.org/m/pr6wmdeq). Both problems were designed to engage the students in reasoning with reflectional symmetry and the idea that the shortest path between two points is a straight line.

Supporting Students in Their Journey to Think Structurally

This paper demonstrates how a teacher can use a sequence of mathematical problems that consist of an anchoring problem and its horizontal and vertical extension problems to progressively move students from recognizing a structure to perceiving its generality and then to reasoning with the general structure. When monitoring the work of students in small groups or individually, the teacher aims to support students to recognize an intended structure or elements of it introduced in an anchoring problem. In whole-class discussions, the teacher carefully selects and organizes students' responses to the anchoring problem and connects their methods with the intended structure. After solving an anchoring problem, students can be provided the opportunity to solve a set of its horizontal extension problems. In doing so, students are guided to grasp the generality of the structure. The use of vertical extension problems provides students opportunities to reason with the general structure. It further reinforces students' dispositions and abilities to think structurally. Through such carefully facilitated problem-solving experiences, students develop their dispositions and abilities to look for and recognize structures, to probe into and act upon structures, and to reason with general structures.

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