Bridging Math Content and Teaching through Pre-Service Undergraduate Apprenticeship

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Abstract: *Two undergraduate mathematics majors worked with a university mathematics educator in a mathematics course for elementary teachers to help future secondary math teachers. They attended weekly meetings and class sessions to deepen their knowledge of teaching math. This article describes five activities used to support the apprentices' learning, including recording teacher questions, switching roles, leading warm-ups, and requesting and responding to student feedback. These activities focused on foundational math concepts learned in grades PK-8.*

Keywords: *Pre-service teachers, apprenticeship, elementary education, statistics, content knowledge*

Introduction

Mathematics majors preparing for a single subject credential typically are not required to earn credits for elementary and middle school mathematics content courses (Conference Board of the Mathematical Sciences, 2012). Yet teaching secondary mathematics and statistics necessarily builds on foundational K-8 mathematical concepts. In Ball and colleagues (2005), conceptualization of *mathematical knowledge for teaching* (MKT), which can be described as "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill, Rowan, & Ball, 2005, p. 373), one component of the six-part egg model is called horizon content knowledge (HCK). Ball and Bass (2009) describe HCK as "a kind of mathematical 'peripheral vision' needed in teaching, a view of the larger mathematical landscape that teaching requires" (p. 1). Mosvold & Fauskanger (2014) describe HCK as when a teacher has "a sense of the mathematical environment surrounding the current location of their instruction." Often given less attention in studies of mathematical knowledge for teaching (e.g., Mosvold & Fauskanger, 2014), it is this horizon way of knowing for future teachers that we sought to deepen throughout this study.

To support two undergraduates intending on teaching secondary mathematics in learning foundational K-8 mathematics, Negrelli and Ortiz apprenticed with university mathematics educator Druken in a mathematics for elementary teachers course. This experience was part of a Robert Noyce National Science Foundation (NSF) grant, *Transitioning Mathematics Majors to Teaching*. In this article, we share five practical activities from our experiences that can be used with undergraduate mathematics major apprentices to examine complexities of teaching mathematics during a mentored apprenticeship, and particularly to gain 'a view of the larger mathematical landscape' to support their mathematical teaching careers.

Context

As part of the NSF-funded Robert Noyce Teacher Scholarship Program, Druken mentored undergraduates Negrelli and Ortiz, starting in January 2022. Both undergraduates were completing their final semester of a bachelor's of arts degree in mathematics with a teaching concentration. Druken was not directly supported by the grant except to provide mentoring to two apprentices for one semester. This was the fourth semester Druken mentored mathematics apprentices.

Negrelli & Ortiz were assigned to different sections of a mathematics department course, *Fundamental Concepts of Elementary Mathematics*, both taught by Druken. This 15-week course met weekly for two 75-minute sessions, virtually for the first two weeks and in-person for the remaining 13 weeks. Beyond attending all class meetings, Negrelli, Ortiz, and Druken met virtually each week to discuss experiences across class sections. Meeting notes were recorded to capture apprentices' weekly assignments and their responses, including reflections and evidence from class. These notes served as the main data source for this article.

Weekly meetings formed the backbone of a reflective structure for inquiring into mathematics teaching practices. Meetings were designed to be safe learning spaces for introspection and exploration with an experienced mentor to guide the conversation and provide support. While meeting topics varied, the team often discussed upcoming activities, engaged in conversations about equitable instruction, focused on language used during instruction to ensure a sense of belonging (e.g., recognizing the use of 'guys' versus 'mathematicians' when collectively referring to the class), reflected on observations of students in the classroom (e.g., one week's activity was to follow and connect with three different students to get to know them as individuals and mathematicians), and summarized and responded to student feedback. Since Negrelli and Ortiz apprenticed in different sections with Druken, everyone learned from one another's experiences while sharing events from their section. These topics will be discussed below in greater detail.

Druken has experience researching, facilitating, and observing lesson studies (Druken, 2015; Druken, Marzocchi, & Brye, 2020; Druken, 2022). Influenced by cycles of studying, co-planning, co-teaching, and debriefing that comprise lesson study, these features inspired interactions with and learning opportunities for apprentices. To focus apprentices' attention during class and guide debriefing conversations during meetings, Druken chose weekly topics based on the interests of the apprentices, with their input. In this article, we use *students* to refer to enrolled members of the university course; *apprentices* to refer to undergraduate students Negrelli and Ortiz; and *instructor* to refer to the course faculty instructor of record Druken. We use *team* to refer to the learning community formed by the instructor and apprentices.

The following activities were identified by the apprentices as memorable, important, impactful, and beneficial for preparation in their math teaching credential programs.

Results

Experiencing Foundational Mathematics as Mathematics Majors

The apprenticing course, optional for mathematics majors at our institution and the second in a two-course series, unpacks core concepts to the K-8 mathematics curriculum and is typically taken by undergraduates intending to earn a multiple-subjects teaching credential. Course topics include the real number system, geometry, probability and statistics, and problem solving. Content focus for each week (minus assessment in weeks 7 and 14) during the study period included:

1. Introduce Statistical Variability;
2. Problem Solve with Statistics;
3. Represent, Analyze, & Interpret Data;
4. Summarize & Describe Distributions;
5. Draw Inferences about Populations;
6. Chance with Probability;
7. Measure with Standard & Non-Standard Units;
8. Reason Spatially;
9. Argue about Geometric Shapes;
10. Justify Area & Perimeter;
11. Explain Surface Area & Volume;
12. Transform Shapes & Objects; and
13. Putting it All Together.

While providing details on course design is beyond the scope of this paper, many topics were inspired by frameworks such as the *Guidelines for Assessment in Instruction for Statistics Education II* (GAISE II) (ASA, 2020), the *Statistical Education of Teachers* (SET) (ASA, 2015), and the eight mathematical practices within the *Common Core State Standards* (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010).

Apprentices experienced elementary mathematics and statistics topics that support secondary mathematics, which is the focus of their intended credential. For example, they learned that data analysis standards are assessed as early as the first grade, and that these include organizing, representing, and interpreting data (CCSSI, 2010). One apprentice stated that learning about representing data, particularly through the use of technologies such as the *Common Online Data Analysis Platform* (CODAP), was particularly meaningful. Another important foundational idea learned by apprentices centered on misleading graphs. The apprentices found it powerful to apply data display frameworks to their every day lives, finding many public examples of graphs that were arguably misleading.

As part of learning statistics and geometry, apprentices learned several technologies to support students' mathematical learning. We used CODAP (http://codap.concord.org), a free web-based data science tool designed to support student learning in grades 6-14. CODAP assists in the analysis of data sets, including pre-uploaded CODAP datasets, datasets found online, or any .csv files collected through a survey. Additionally, *Desmos Classroom* (https://teacher.desmos.com) and *Geogebra* (https://www.geogebra.org/) were used to engage students in algebraic and geometrical thinking. Apprentices learned firsthand how these technologies could be used in a secondary setting. Additional examples of mathematical topics useful for training PK-8 teachers of mathematics can be found in Druken & Frazin (2018) on the topic of modeling using math trails and Druken & Marzocchi (2019) on the topic of fractions.

Recording Questions Posed by Instructor

A powerful moment occurred when apprentices were invited to record all questions posed by the instructor across a week's worth of classes during a statistical problem solving unit. Before continuing on, the reader may want to watch Annie Fetter's 2011 Ignite Talk, "Ever Wonder What They'd Notice?" (https://www.youtube.com/watch?v=a-Fth6sOaRA) from the National Council of Teachers of Mathematics Annual Conference. The techniques highlighted in the video were employed frequently by Druken during the unit. Apprentices recorded all questions verbally posed by Druken during two class sessions, analyzed them, then summarized findings during a weekly meeting. The following groups were formed based on the intended purpose of the questions as reported by the instructor. While student responses are intentionally not included here, we leave it to the reader to speculate how their students might respond to the following questions.

To Cultivate Classroom Culture

- Turn to someone near you and share one thing about your weekend.
- What is one thing you wish your instructor knew about learning via Canvas (a learning management system) or being a student in general this semester?

To Check for Student Understanding

- What did you notice and wonder about the data? What's interesting about this data?
- Is the question clear of what I am asking? Does that make sense?
- On a finger scale from 1 to 5, how confident are you with computing *mean absolute deviation* (MAD)?
- Can someone tell us what you were noticing?

To Grow Skills

- What kind of data could we collect from our future students? Is your example attribute categorical or numerical? Discrete or continuous?
- Do I read the level of the graph from the top of the lid or the top of the can (on a misleading pictograph involving a garbage bin)?
- Does it look like the graph has a consistent scale on the y-axis?
- How do I write "31.5 million" without words, just digits?

To Progress Concepts

- What are some common measures of center? What about measures of spread?
- What concept do we want to think about when we think of *mean* (average)?
- What does it mean to 'tidy up the data'?
- Do you think anything is misleading about this graph?
- When do you think probabilistic thinking is useful in the real world?
- How would you visualize a mean absolute deviation of 0?
- I'm wondering if data can be both categorical and numerical?

To Encourage Reflection, Conversations, and Collaborations

- Could you say more?
- Turn to someone and share what you notice: What story is being told with the graph?
- Draw an inference based on the graph or share what you heard your classmates say
- Can anyone explain how CODAP helped or hindered their learning?
- What was your big takeaway from (topic)?

Recording the instructor's questions served as a useful learning experience for the team. The activity gave Druken an opportunity to reflect on questioning practices, something challenging to notice in-the-moment. The activity also gave apprentices a chance to understand that questions serve different purposes and can elicit varying levels of information from students.

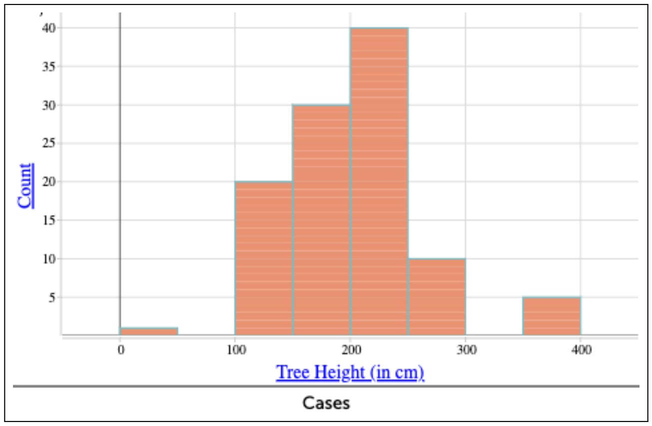
Moreover, university faculty often look for opportunities to engage students in research. Through the analysis of instructor questions, apprentices gain insight into the field of mathematics education and ways in which researchers explore mathematical teaching and learning. This study suggests an approach faculty can use with students to conduct undergraduate research that leads to publications in high-quality refereed journals or to the dissemination of findings at local, regional, or national conferences.

Switching Roles

"I'll pretend to be the student and you the teacher." This is an example of an active learning strategy referred to as switching roles. We piloted it across several class sessions. In this activity, a student acts as the instructor and the instructor acts as a student. Since students in this study mainly consisted of future elementary teachers who planned to earn multiple subjects credentials, we wanted to provide students with the opportunity to put themselves in the position of an educator engaging her students in the practice of 'doing mathematics.'

One example of switching roles might occur as students learn how to interpret graphs. Imagine an instructor projecting a histogram displaying the height of trees consisting of eight bins of width 50 cm each (see Figure 1). Note that values of each bin are left inclusive and right exclusive, with the last bin being right inclusive. The instructor might ask students to share what they notice and wonder prior to analyzing the graph. An important distinction happens when students distinguish between the *number of items in each bin* (for example, 10 trees have a height between 250 cm and 300 cm) with *total number of bins* (eight bins or intervals) while interpreting histogram graphs.

Figure 1: *Histogram of tree heights that could be used to switch roles between teacher and student.*



A question that might prompt switching roles is, "What would you tell your elementary school student if they said, 'there were six trees total' after interpreting this histogram graph?" The ensuing conversation between an instructor and student might go something like this:

COLLEGE STUDENT : [*who has now been invited to wear a teacher hat*] What makes you believe that there are only six trees?

INSTRUCTOR : [*playing along as the imaginary student*] I count one-two-three-four-five-six trees! [*Gestures to six categories along x-axis*]

COLLEGE STUDENT : Okay. Why do you think each bar has a different height? Why might they have different sizes?

INSTRUCTOR : [*reads the y-axis and connects height of the bin as meaning the total number of trees for each specific bar*] Oh wait. Does this mean there are 10 trees in this category? [*points to 250-300 cm*]

The entire class has the opportunity to learn questioning skills while interpreting a histogram graph in the context of making sense of data.

The switching roles strategy supports students, apprentices, and instructors alike in understanding another person's mathematical perspective. Rather than focus on 'wrong ideas,' switching roles helps the instructor to highlight certain features of a graph and invite others to experience mathematical thinking from another person's perspective. The instructor can learn from the student's perspective, while the class learns from their classmate and apprentices learn from students and the instructor. While reflecting on this role play, apprentices emphasized the importance of wanting student contributions to be valued regardless of their ultimate accuracy.

Leading Warm-Ups to Learn Content & Tool Use

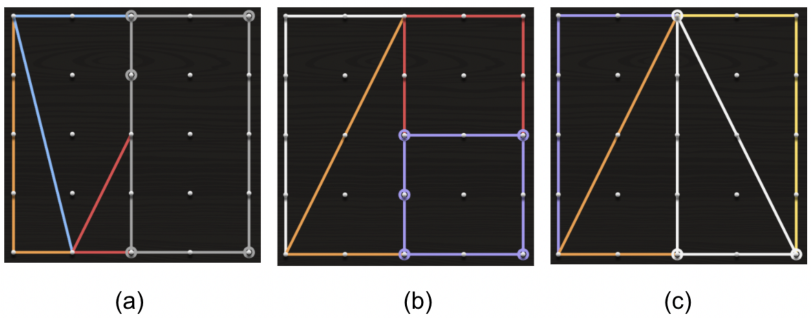
Both teaching apprentices co-planned and individually taught several classroom warm-up activities. An example warm-up is described below.

Geoboard Area of 20 Squares

Manipulatives, both physical and virtual (e.g., https://apps.mathlearningcenter.org/geoboard/), can be used to provide students with concrete building experiences related to mathematics. One example is a physical geoboard, a flat board with an array of pegs that are used with elastic bands to create shapes and designs. Apprentices facilitated an activity that involved creating shapes with physical geoboards to initiate discussion of area formulas. Prior to the activity, students explored characteristics of triangles and quadrilaterals, including defining characteristics and definitions.

The geoboards used in this study contained 25 pegs. Assuming the smallest square has area of 1 square unit, the area of the entire board is 16 square units. In a previous class, Druken invited students to first partition their board into four parts, then four equal parts, and to reflect on any differences between the two creations (see Figure 2). The goal was to recognize that when conceptualizing fourths using an area model, unit fractions must have the same area but may be non-congruent shapes. Druken then asked students to create a rectangle using only two triangles and calculate the area of one triangle. Students were able to determine the triangle's area was half of the base times height since the rectangle (which is the same as the base times height) was twice as large as one triangle.

Figure 2: *Geoboards showing (a) four parts, (b) fourths with non-congruent pieces, and (c) fourths with congruent pieces.*

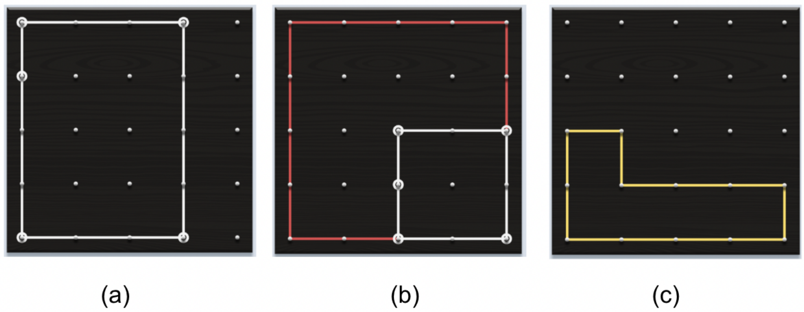


Each apprentice asked students to create a non-rectangular shape with an area of 20 squares. Since the geoboard only had an area of 16 square units, we anticipated students would determine that it wasn't possible to create such a shape with our boards. In the beginning, students exchanged confused looks as we attempted to not provide any hints as to a solution to the task. As more time was allotted for exploration, students continued to grapple with the posed question.

We were surprised when students responded creatively when designing a non-rectangular shape with 20 squares. One student changed their unit of measurement from a continuous to a discrete context and found 20 *pegs* instead of 20 *squares* (see Figure 3a). This allowed us to highlight ways in which area connects to multiplication as an array and ways that area differs. Since area requires the counting of square units while an array involves the counting of discrete objects, the units that are enumerated in each context differ. In the discrete array case, there were 20 pegs and in the continuous area case, there were 20 square units.

Another student created two overlapping shapes, one 4×4 square and one 2×2 square, and counted 16 and 4 square units (see Figure 3b). This allowed us to think about the definition of non-rectangular and whether a shape can consist of overlapping components. A third student changed the scale of what constituted a unit, partitioning each original unit into four smaller squares. They then outlined five original units in the shape of a hexagon, which had an area of 5 original units $\times$ 4 small units/original unit = 20 small units when rescaled (see Figure 3c).

Figure 3: *Geoboards showing (a) 20 pegs instead of 20 square units , two overlapping shapes, (b) a 4x4 and 2x2 square, whose areas sum to 20 square unit, and (c) a hexagon made of five large squares, where each square is composed of 4 small square units.*



With the help of the mentor faculty member, teachers can navigate a 'wrong answer' by asking questions or by looking for what is correct about the solution. For example, we learned that some students saw the two quantities 'number of pegs' and 'side length of a polygon' as interchangeable, which affected how they thought about the activity. Apprentices were able to recognize where students saw twenty units (as 20 discrete pegs rather than 20 continuous squares of area) and to reiterate area instead as a process involving covering a two-dimensional space. This also served to distinguish between *discrete* quantities, or separate non-touching objects that can be counted, and *continuous* quantities, those that can only be measured and whose value might change if using a more precise measuring tool.

Seeing students flexibly change the scale of the geoboard or imagine overlapping squares provided us with the opportunity to experience students' creativity. As one apprentice reflected, "if we jump in right away to explain, we reduce the room they have to think differently." Apprentices learned that when providing students with space to arrive at their own solutions rather than be given the 'correct' solutions right away, educators can improve the intent of their original question and consequently provide a richer learning environment for all. By placing value on multiple ways to creatively solve a problem, one where even the instructors did not anticipate all ways students might interpret and answer the question, instructors can make students feel welcomed and an integral part of their own learning experiences.

Requesting & Responding to Student Feedback

Several opportunities allowed students to share feedback with apprentices and the instructor throughout the course. These data were requested to systematically understand students' experiences learning mathematics, respond to students' needs, and inform instruction.

Student Needs Survey

An online survey provided throughout the first week of class was the Student Needs Survey (link to Survey 1; link to make a copy of Survey 1). The purpose of this survey was to collect information on their needs to be successful while learning mathematics and their views on mathematics classrooms. Other questions invited students to finish the sentence: "I really like when my mathematics classes are ...", "I feel supported by my mathematics teacher when ...", "I feel discouraged by my mathematics teacher when ..."

Checking for Understanding

One way to check for understanding includes taking pauses. For example, the instructor might say, "Turn to someone and explain in your own words what we just talked about. You have 30 seconds—go!" By asking students to turn to a partner, triad, or group to engage in a mathematical conversation, we provided them with opportunities to process and question mathematical concepts and procedures while the instructor listens for challenges.

A second way we checked for understanding included regularly asking reflective questions and using routines like warm-ups such as graph talk routines (Marzocchi, Turner, & Druken, 2018), slow-reveal graphs (Laib, n.d.), or weekly writing prompts to support students in making connections between mathematics concepts and their lived experiences.

A final way to check for understanding involved gathering class-based data to analyze using CODAP. Based off of an assignment where students created statistical and survey questions, the instructor built a start-of-semester class survey (link to Survey 2; link to make a copy of Survey 2). Survey questions asked about students' lives, such as commute times (one way), favorite location on campus, type of protein typically eaten, type of caffeinated beverages typically drank, etc. The instructor might create one graph using CODAP to share at the start of class as a warmup, asking students to analyze it. This allowed the class to make claims about aggregated categorical and numerical displays of univariate and bivariate data while learning about their peers.

We recommend those interested in learning about additional assessment techniques to explore Feldman (2018) on why grading for equity matters and Keeley and Tobey (2011) that provides 75 practical ways for linking instruction, assessment, and learning in mathematics classrooms.

Mid-Semester Feedback

Another way to request and respond to student feedback occurred through mid-semester feedback. Students were asked to anonymously share responses to two questions: *What is something that is working well for you? What is something that could be changed or improved?* The instructor then asked apprentices to summarize student responses and present findings at the weekly meeting. Student responses previously integrated into the course include: providing study guides for exams, locating additional practice problems, and presenting solutions to all quiz problems in addition to posted solutions.

Conclusion

Through attending all class sessions and weekly professional development meetings, apprentices were able to learn foundational K-8 mathematics to support grades 6-12 mathematics; record questions posed by the instructor; experience switching roles between instructor and mathematics student while facilitating learning; lead warm-up activities to learn mathematics content and pedagogical strategies; and request and respond to student feedback. We argue that when undergraduates mathematics majors apprentice in courses for future elementary mathematics teachers, they experience rich opportunities for learning how to teach mathematics, which are vital for secondary teachers of mathematics. In particular, secondary teachers can better support their students in a way that builds from mathematics recently learned in elementary and middle school grades.

We hope these concrete activities provide insights and spark curiosity on how apprenticeships can engage future secondary mathematics teachers in examining the complexities of learning to teach mathematics, particularly through deepening their horizon content knowledge across grades PK-12.

Acknowledgements

- This material is based upon work supported by the National Science Foundation under Noyce Grant #1758389.
- Our institution sits on the traditional territory of the Kizh. We thank them for our continued use of the natural resources on their ancestral homelands.

References

American Statistical Association (ASA). (2020). *Pre-K through 12 guidelines for assessment and instruction in statistics education (GAISEII): A framework for statistics and data science education.* https://www.amstat.org/education/guidelines-for-assessment-and-instruction-in-statistics-education-(gaise)-reports

American Statistical Association (ASA). (2015). *Statistical education of teachers (SET) report.* https://www.amstat.org/publications/education-publications

Association of Mathematics Teacher Educators (AMTE). (2017). Standards for Preparing Teachers of Mathematics. http://www.amte.net/standards

National Governors Association Center for Best Practices and Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics (CCSSM).* Washington, DC: Authors https://learning.ccsso.org/common-core-state-standards-initiative

Conference Board of the Mathematical Sciences. (2012). *The Mathematical Education of Teachers II (MET II).* Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.

Druken, B.K. (2023). Practicing teachers' perspectives on the purposes of mathematics lesson study. *Journal of Mathematics Teacher Education and Development*, *25*(1), 1-22. https://mted.merga.net.au/index.php/mted/article/view/649

Druken, B.K. (2015). Social capital, social networks, and lesson study: Sustaining mathematics lesson study practices. In Bartell, T.G., Bieda, K.N., Putnam, R.T., Bradfield, K., & Dominguez, H. (Eds.), *Proceedings of the 37th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (p. 663–670)*, East Lansing, MI: Michigan State University.

Druken, B.K. & Frazin*, S. (2018). Modeling with math trails. *Ohio Journal of School Mathematics*, *79*, 43–53. https://library.osu.edu/ojs/index.php/OJSM/article/view/6282

Druken, B.K. & Marzocchi, A.S. (2019). Strings, bears, & boards: Making fractions meaning-filled. *Ohio Journal of School Mathematics.* https://library.osu.edu/ojs/index.php/OJSM/article/view/7209

Druken, B.K., Marzocchi, A.S., & Brye, M.V. (2020). Facilitating collaboration between mathematics methods and content faculty through cross-departmental lesson study. *International Journal for Lesson and Learning Studies*, *10*(1), 33-46.

Feldman, J. (2018). *Grading for Equity: What It Is, Why It Matters, and How It Can Transform Schools and Classrooms*. Corwin: Thousand Oaks, CA.

Keeley, P. & Tobey, C.R. (2011). *Mathematics-Formative Assessment: 75 Practical Strategies for Linking Assessment, Instruction, and Learning.* Corwin & NCTM: Thousand Oaks, CA.

Laib, J. (n.d.). Slow reveal graphs: An instructional routine to promote sense making about data. Available online at https://slowrevealgraphs.com/

Marzocchi, A.S., Turner, K., & Druken, B.K. (2018). Using graph talks to engage undergraduates in conversations around social justice. *Problems, Resources, and Issues in Mathematics Undergraduate Studies.* https://doi.org/10.1080/10511970.2018.1456499

Mosvold, R.,& Fauskanger, J. (2014). Teachers' beliefs about mathematical horizon content knowledge. *International Journal for Mathematics Teaching & Learning.*

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