Walking the Ellipse

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Abstract: *The Walking the Ellipse activity is a mathematical modeling task that allows students to explore the ellipse in a plausible real-world scenario. Students will gain a better understanding of the definition of the ellipse by creating an elliptical model first using markers, paper, and string and later dynamic geometry. This activity was successfully performed with strong middle grades students in a summer camp and can be modified to be performed with students at their school.*

Keywords: *Ellipse, conic sections, mathematical modeling, geometry*

Introduction

Modeling in mathematics can be seen as one of the most important skills that a mathematics student can learn. Indeed, from the Common Core Standards for Mathematics, "Making mathematical models is a Standard for Mathematical Practice" (NGA Center & CCSSO, 2010, p. 57). Further, modeling draws connections among mathematical topics and "[L]inks classroom mathematics and statistics to everyday life, work, and decision-making" (NGA Center & CCSSO, 2010, p. 72). Also, creating geometric models using physical materials and using technology is seen as a valuable endeavor since students are expected to "Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.)" (NGA Center & CCSSO, 2010, p. 76). Finally, research has suggested several links among successful problem-solving, students' positive affect, and metacognition. Williams & Huang (2015) write: Metacognitive knowledge facilitates the employing creative problem-solving strategies: identifying the key concepts underlying the problems, recognizing types of representations that could be useful, and carrying out operational processes to finding solutions. A good problem solver who has developed metacognitive habits can reflect on their problem-solving activities, monitor and regulate their strategies, and justify their solutions (Williams & Huang 2015, Section 2.2).

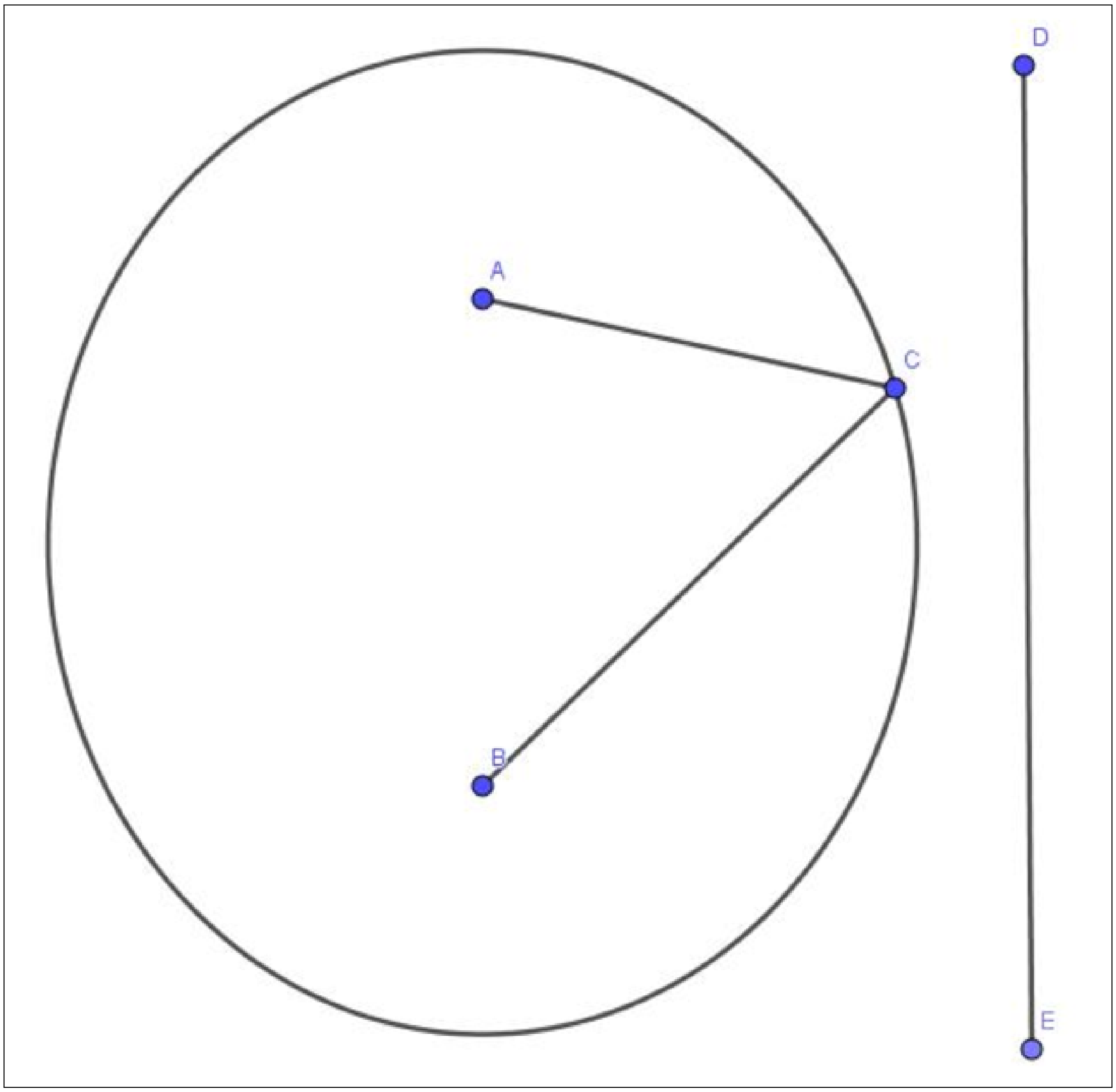
Conic sections provide a rich area for modeling for the mathematics student given their applications to real-world phenomena. Indeed, parabolas, hyperbolas and ellipses have applications in optics, radio and satellite systems, woodworking, and even medicine ("Conics in Real Life," 2020; Almukahhal et. al., 2022; Sloggatt, 2012; Glydon, 2023). However, synthetic definitions of conic sections as collections of points can be seen as esoteric and leave students with misunderstandings about their nature.

In this article, an activity is presented which can allow students to model a real-world situation with an ellipse in a way that can bring understanding to its definition. The activity can be performed with middle or secondary grades students having only a background in ratio and proportion.

Learning Goals of the Stolen Loot Activity

An ellipse is defined to be the collection of all points in a plane, the sum of whose distances from two fixed points is a fixed constant. For example, If A and B are foci (fixed points) of the ellipse and the fixed constant distance is the length of segment $\overline{ED}$, then an ellipse can be created as in Figure 1. Note that $m(\overline{AC}) + m(\overline{CB}) = m(\overline{DE})$ and $m(\overline{ED}) \geq m(\overline{AB})$.

Figure 1: *Ellipse with foci A and B and constant length.*



While correct, this definition is elusive to even some of the strongest students. In this activity, students will create a geometric model based on a plausible real-world scenario involving stolen money (loot). They will recognize an ellipse by exploring its definition in a tangible way by creating a physical model using materials. They will then walk the ellipse created on a map to find the stolen loot. Finally, they will explore the ellipse using dynamic geometry to solidify their understanding and realize connections between the ellipse and the real-world.

Description of the Stolen Loot Activity

In this activity, students are placed in small groups of 4-5 students and are introduced to a forensic situation in which they are to solve a robbery by recovering stolen loot (fake money), hidden on their school campus. The groups are kept small to ensure collaboration among group members, and some natural roles within the group might emerge. When this activity was performed with small groups, it was common that some students took on the role of calculation expert, while others took on roles relative to the materials provided to perform the activity. The activity consists of four parts: (1) Reading the activity in detail, noting the important information, (2) modeling the situation described in the reading using physical materials, (3) recovering the stolen loot based on the model, and (4) analyzing the activity using dynamic geometry to solidify their understanding of an ellipse.

Part 1: Reading the Activity

The narrative serves to frame the situation:

> You live in College Hall at Enormous University and heard that yesterday afternoon the coffee shop on campus was robbed! Witnesses say that a short sandy-haired person was seen with a bag leaving the scene of the crime on a purple mountain bike, exiting from the northern exit of the building housing the coffee shop. This piece of news captures your attention because you happen to own a purple mountain bike, where it is usually kept chained to a bike rack on the northern side of your dorm hall. Further, your roommate is short with sandy brown hair! Just this morning, some friends told you that your roommate was saying that no one would catch the robber as they probably stashed the loot. Being a serious cyclist, you check the odometer on your bike daily to track how much you have ridden. After riding yesterday morning, you recall that the odometer read 654 feet. Checking it today, you see it now reads 2,790 feet, though you have not ridden the bike since! Moreover, you cannot find the key to your bike chain lock, which usually hangs by your door. The police question both you and your roommate and find them with your key in their backpack. They tell the police that you must have slipped the key in their bag to frame them. The police are unsure what to believe but claim that, if the loot were found, trace evidence on the bag might point to the real culprit.
>
> *Your task*: Find the loot to help prove your innocence.

Of course, the scenario itself and the distances could be modified based on the school campus of the students involved in the activity. This would require some planning, but the scenario and distances could be worked out using Google Earth (Gorelick, N. et. al., 2017) or other global positioning system that allows measurements of distances to be calculated. Of important note when planning is that one should be careful about measuring linear distances across terrain that has changes in elevation. It is best to perform this activity on a campus that is relatively flat.

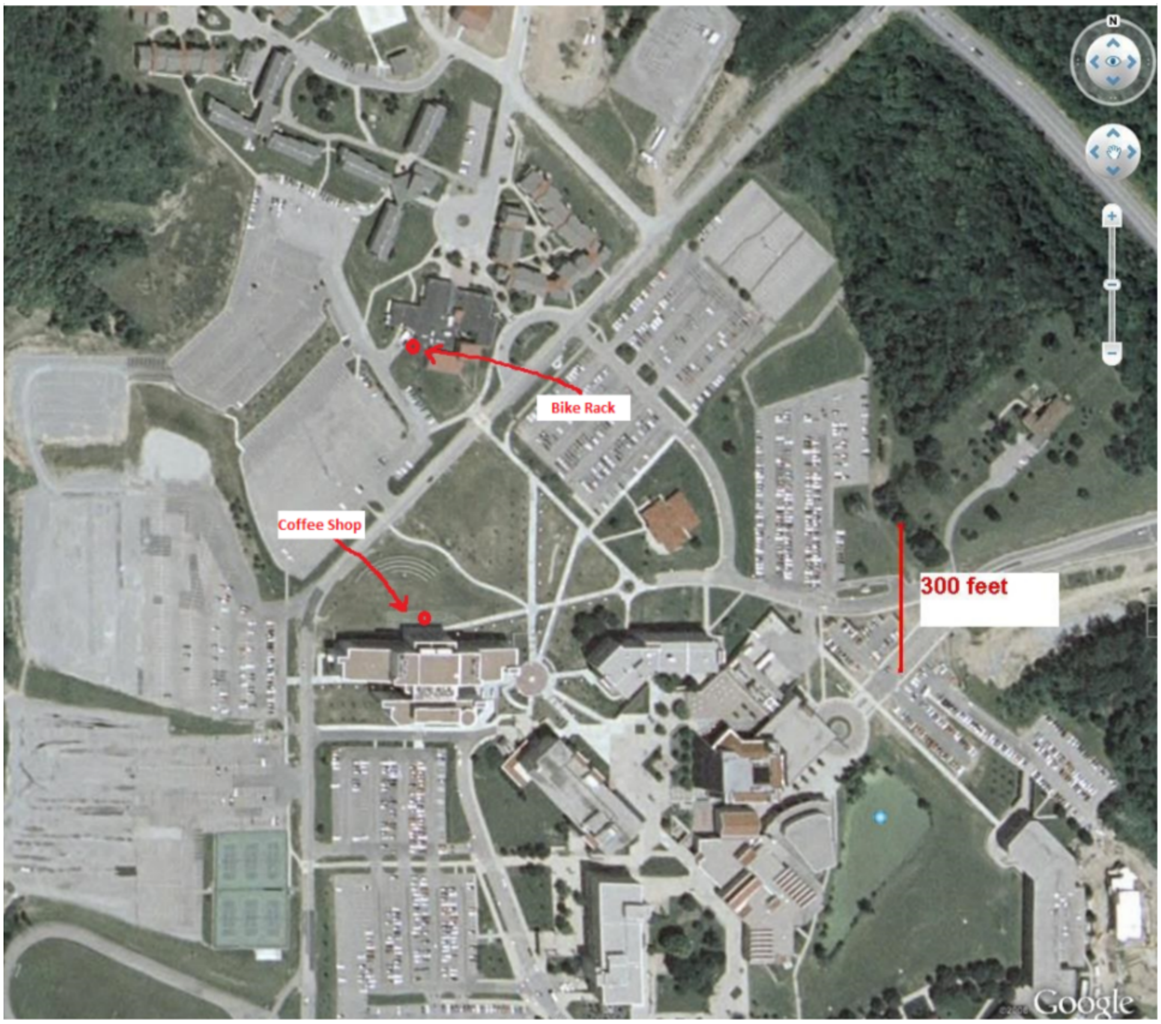
For this task, students are presented with the following tools:

1. Several maps with scales (including one large) of their school campus.
2. Strings, markers, and rulers.
3. Dynamic Geometry System (DGS) such as Geometer's Sketchpad (Jackiw, 1991). Another possibility is to use Geogebra (Geogebra, 2023), which is free.

Part 2: Modeling the Situation

The modeling part of this situation begins with students analyzing a map of their school campus, such as the one illustrated in Figure 2. This map should be larger than pictured (poster-sized), allowing for students to use their tools appropriately.

Figure 2: *Map of school campus.*



So, it can be concluded that the bike was ridden 2,136 feet before being returned to the bike rack. Next, students might infer that the rider rode the bike to the coffee shop, stashed the loot, then returned the bike to the bike rack. So, finding the distance from the bike rack to the coffee shop is important here. Using the ruler and map scale, students can then calculate the scale factor.

300 feet = 3 inches

So, the scale factor would be 300 feet per 3 inches or 100 feet per inch. Of course, this might vary depending on the size of the map used. Measuring the distance from the bike rack to the coffee shop on the map would yield 5.4 inches.

Setting up the proportion using the scale factor, we have

$$\frac{100 \text{ feet}}{1 \text{ inch}} = \frac{d \text{ feet}}{5.4 \text{ inches}}$$

Solving this proportion for d gives $d = 540$ feet. So, the distance from the bike rack to the coffee shop is 540 feet.

Now, it can be inferred that the robber must have ridden $2,136 - 540 = 1,596$ feet from the time they left the coffee shop until the time they returned the bike to the rack. This number is now quite important. Using the scale factor, the following proportion can be set up to determine the distance traveled with respect to the map's scale:

$$\frac{100 \text{ feet}}{1 \text{ inch}} = \frac{1,596 \text{ feet}}{m \text{ inches}}$$

Solving this proportion for m gives $m = 15.96$ inches (or about 16 inches).

Now, using the map, students would like to find the possibilities where the bike rider might have stashed the loot. Students should be given time to investigate the possibilities using the map, the ruler, and finally, the string. In short, if one adds the distance from the coffee shop to the loot to the distance from the loot to the bike rack, that should total about 16 inches on the map or about $1,596$ feet on campus, assuming the bike rider did not ride around before or after stashing the loot, a valid assumption since, presumably, the rider did not want to be caught! An additional assumption that should be considered here is that the bike rider has a mountain bike, so is not constrained to riding along roads or paved pathways but can ride cross-country. Without these assumptions, students would only be able to find a search area, constrained by the curve of possible locations.

Students should be encouraged to look for possibilities. For example, if the bike rider rode 400 feet from the coffee shop (about 4 inches on the map) and stashed the loot, he would have had to ride $1,596 - 400 = 1,196$ feet (about 12 inches on the map) back to the bike rack. Students might use one of the maps to illustrate this as in Figure 3.

Note that this would require careful measuring (and perhaps using multiple rulers) to locate this possible location. Students might then proceed to find multiple locations using this method, in effect "mapping out" several possible loot locations by direct measurement on their maps at scale. It is at this point that students should be encouraged to find multiple loot locations and to make a conjecture about all possible loot locations. The collection of all possible loot locations in this case means the collection of all places the sum of whose distances from the bike rack and coffee shop is $1,596$ feet (or about 16 inches on the map).

Considering circles with centers at the coffee shop and bike rack might help. As an example, students might discover—or be led to discover—that, if the biker rode 400 feet from the coffee shop (about 4 inches on the map), stashed the loot, and rode the remaining $1,196$ feet to the bike rack (about 12 inches on the map), the possible loot locations are in the intersection of two circles: one with radius about 4 inches on the map with center at the coffee shop and one with radius about 12 inches on the map with center at the bike rack. This gives two possible loot locations, as seen in Figure 4.

Figure 3: *Map with possible loot locations.*

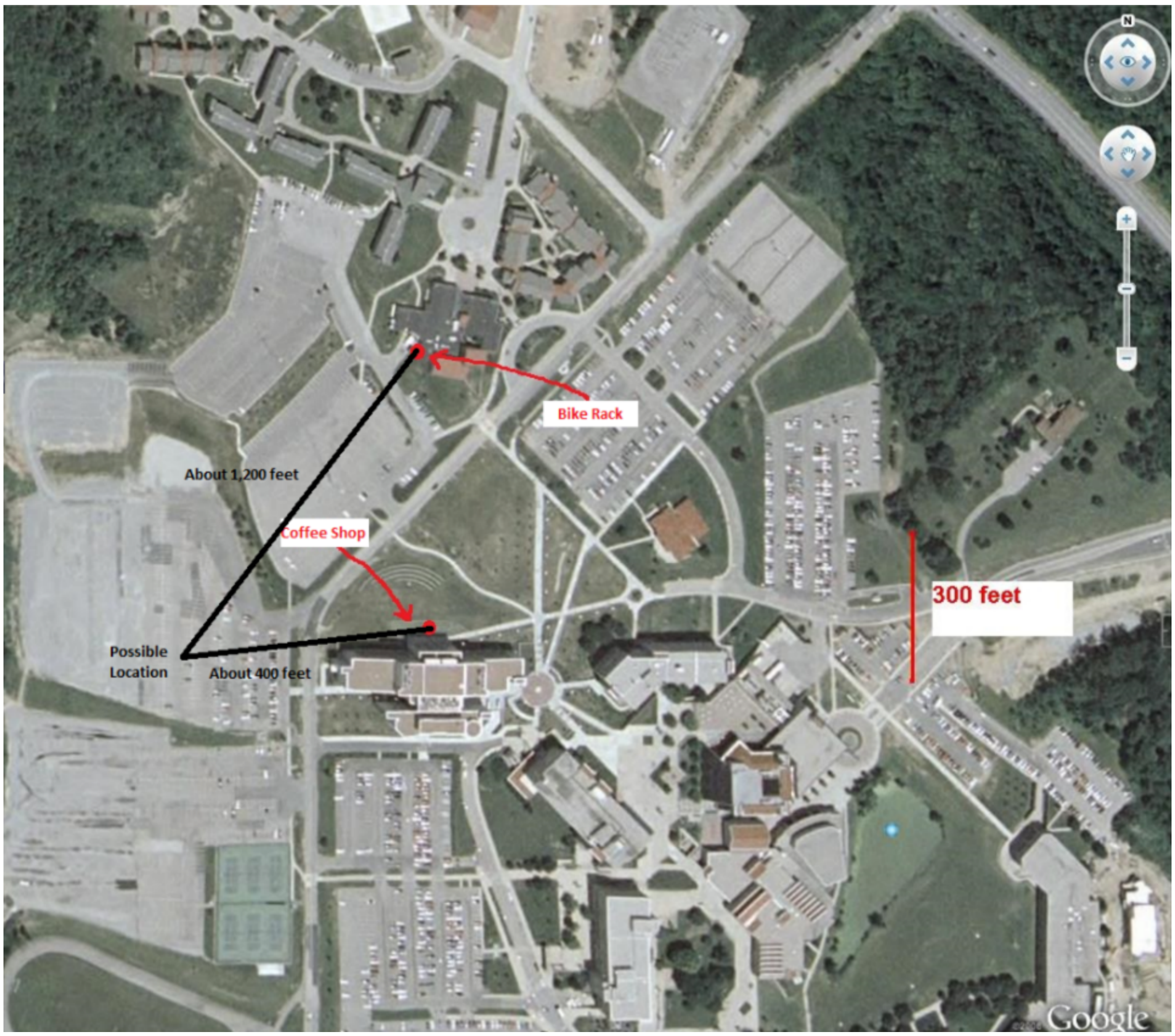
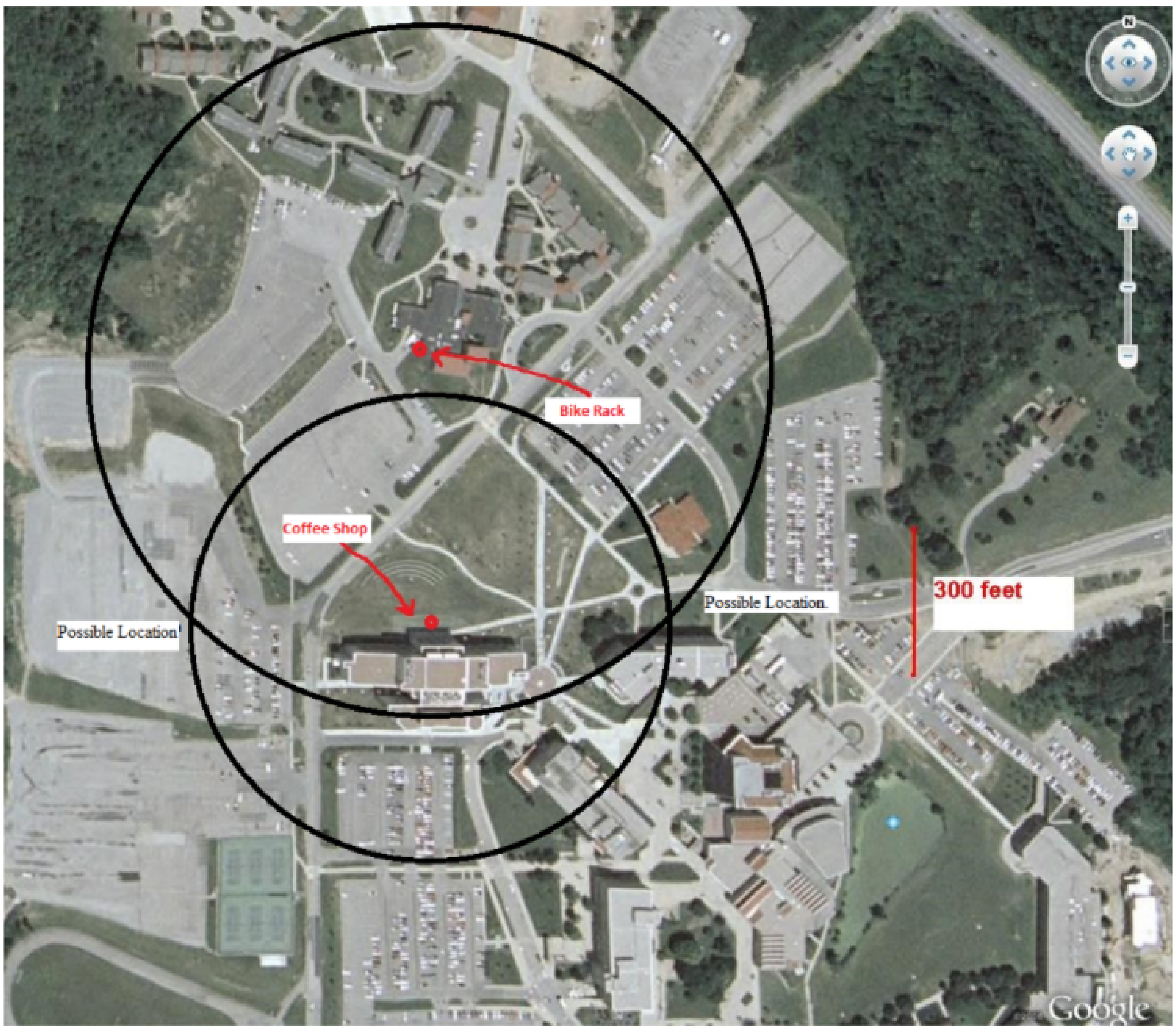
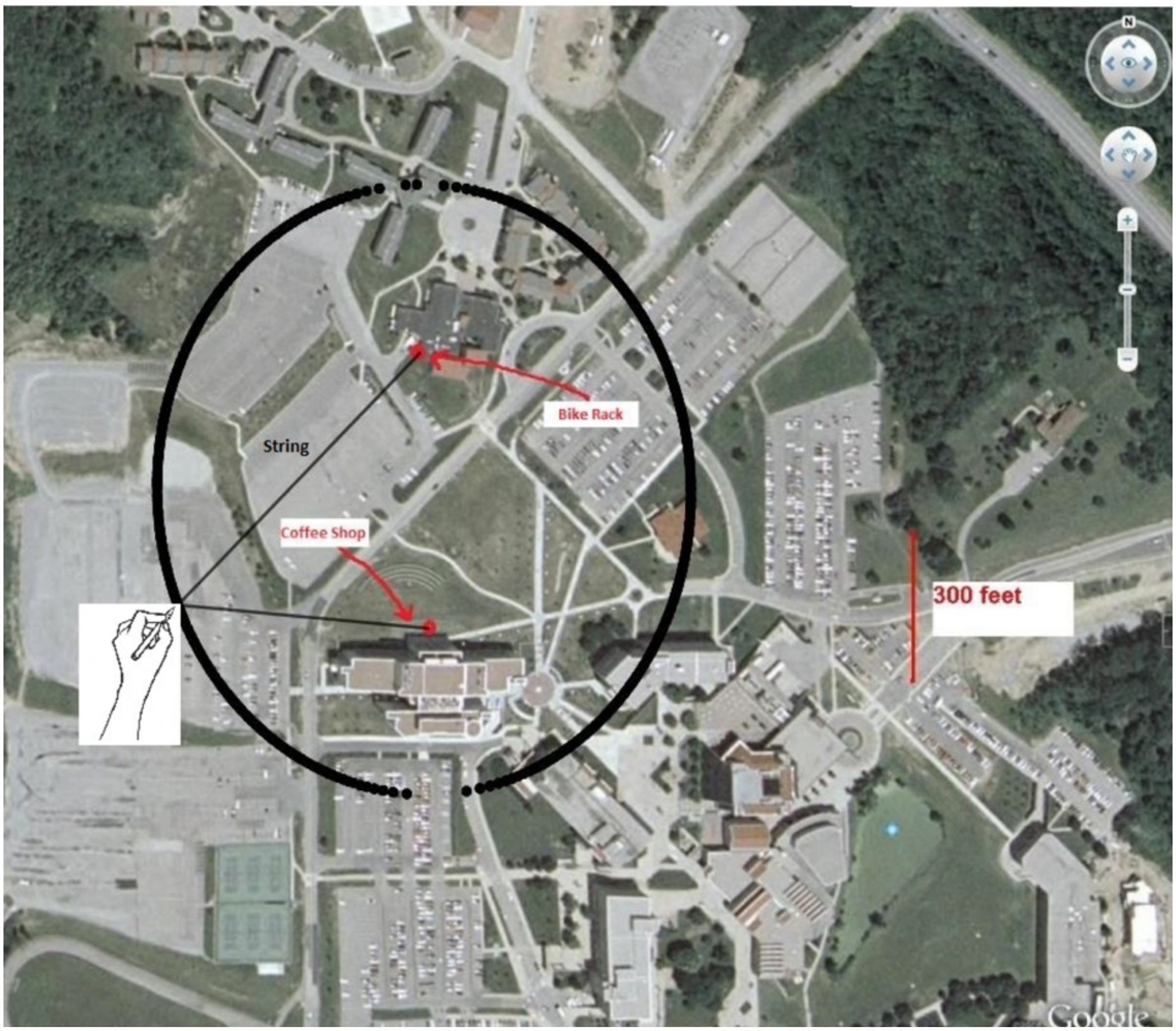


Figure 4: *Map with circles indicating two possible loot locations.*



Some trial and error might result in additional possible loot locations (e.g., the intersection of two circles: one with radius ≈ 6 inches on the map with center at the coffee shop and one with radius ≈ 10 inches on the map with center at the bike rack). Of course, students are looking for all possible loot locations, so students might notice that the sum of the distances from the coffee shop to the loot and the loot to the bike rack is fixed at $1,596$ feet, or approximately 16 inches on the map. With this knowledge, students could then cut a piece of string 16 inches long and use this string on the map to find all possible locations of the stashed loot. They would do this by holding one end of the string at the coffee shop and the other end at the bike rack, while using a writing utensil to trace out the resulting curve in a manner that is relatively well-known (e.g., Apostol & Mnatsakanian, 2011; Shell-Gellash, 2016). This would likely require multiple students to effect. Figure 5 illustrates this.

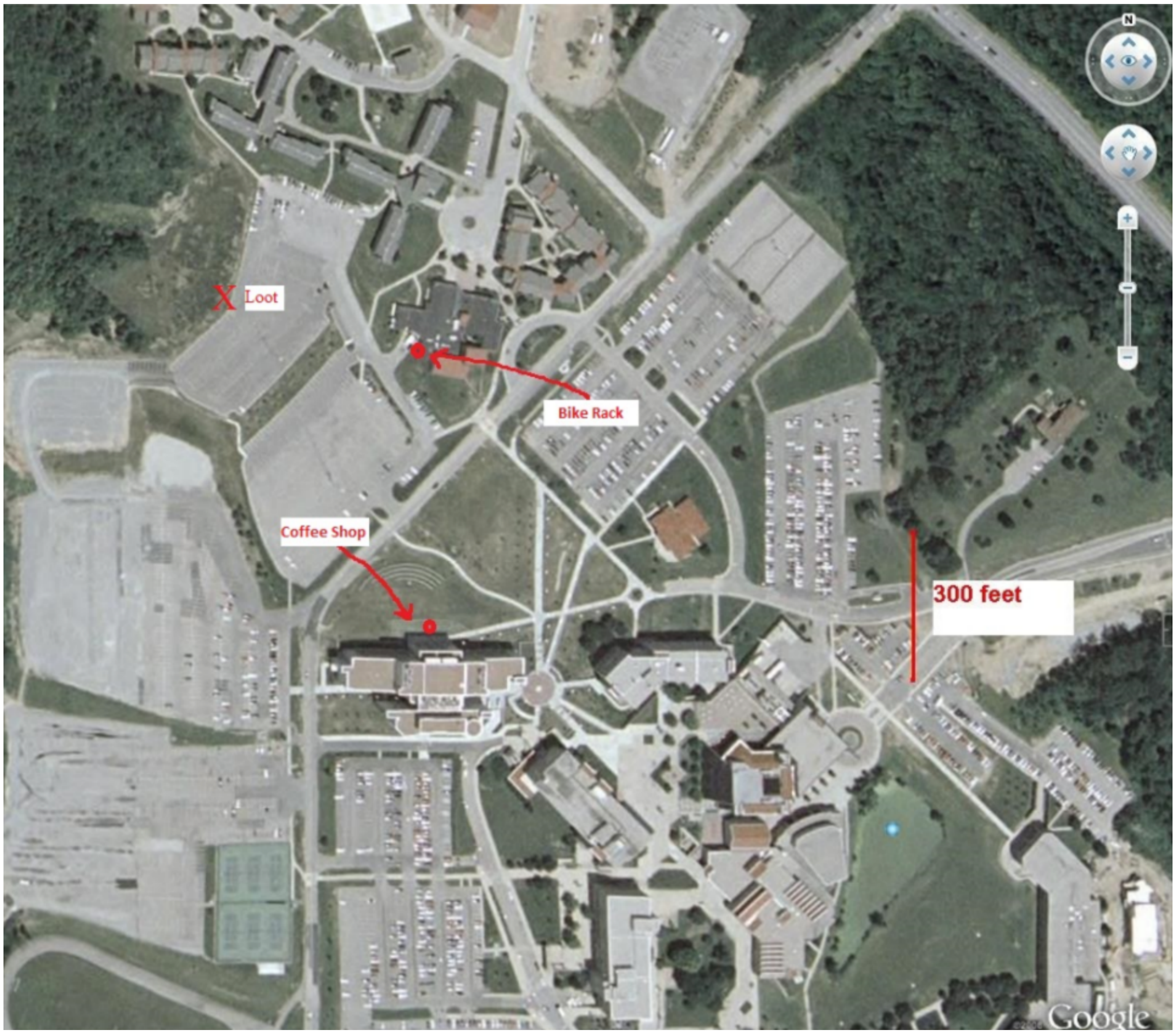
Figure 5: *Map showing all possible loot locations.*



Part 3: Recovering the Loot

Armed with maps with the curve of possible loot locations, students could then search for the missing loot, strategically hidden by the instructor in advance. Using the map and the landmarks indicated, students could then walk the curve of possible locations in search of the stolen loot. In this case, the loot was hidden in a grassy area near a rarely used parking lot. Figure 6 illustrates the location of the missing loot. Note that the loot was hidden along the curve of possible loot locations. When performed with strong middle grades students, it took them approximately 15 minutes to find the loot hidden in a book bag.

Figure 6: *Map showing location of hidden loot.*



Part 4: Analyzing the Activity

After recovering the loot, students will then debrief and review what they have learned from this activity. With some minimal nudging, students can come to realize that the curve here representing all possible loot locations is the collection of all points, the sum of whose distances from two fixed points (the bike rack and the coffee shop) is a fixed constant ($1,596$ feet). This is exactly the meaning of an ellipse with the bike rack and coffee shops being foci (the fixed points) and the $1,596$ feet being the constant (fixed length).

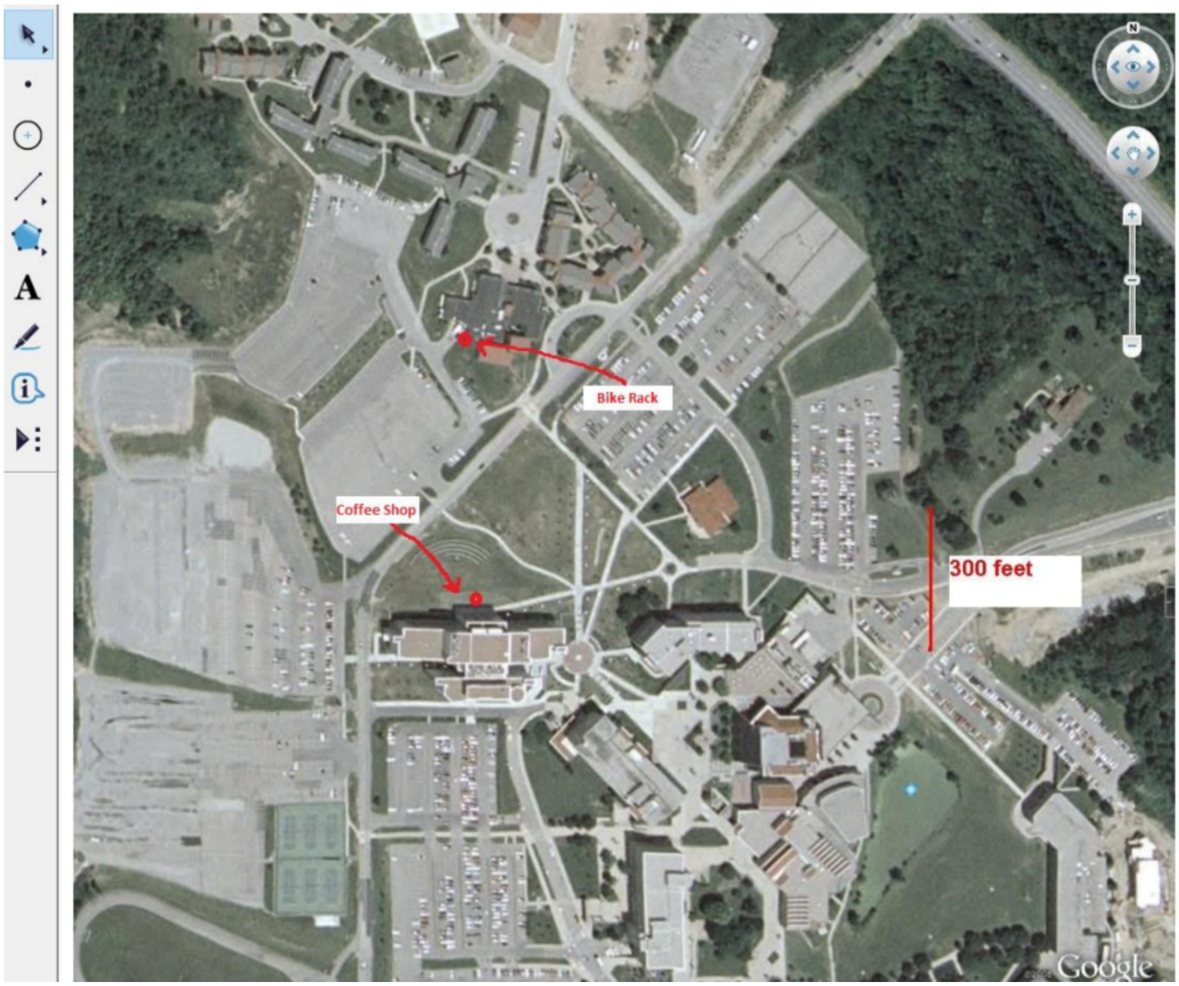
Dynamic Geometry Software

Next, we move into the realm of dynamic geometry, where students can create the model using more advanced technology than markers and string. From the Common Core Standards for School

Mathematics, "Dynamic geometry environments provide students with experimental and modeling tools that allow them to investigate geometric phenomena in much the same way as computer algebra systems allow them to experiment with algebraic phenomena" (NGA Center & CCSSO, 2010, p. 74).

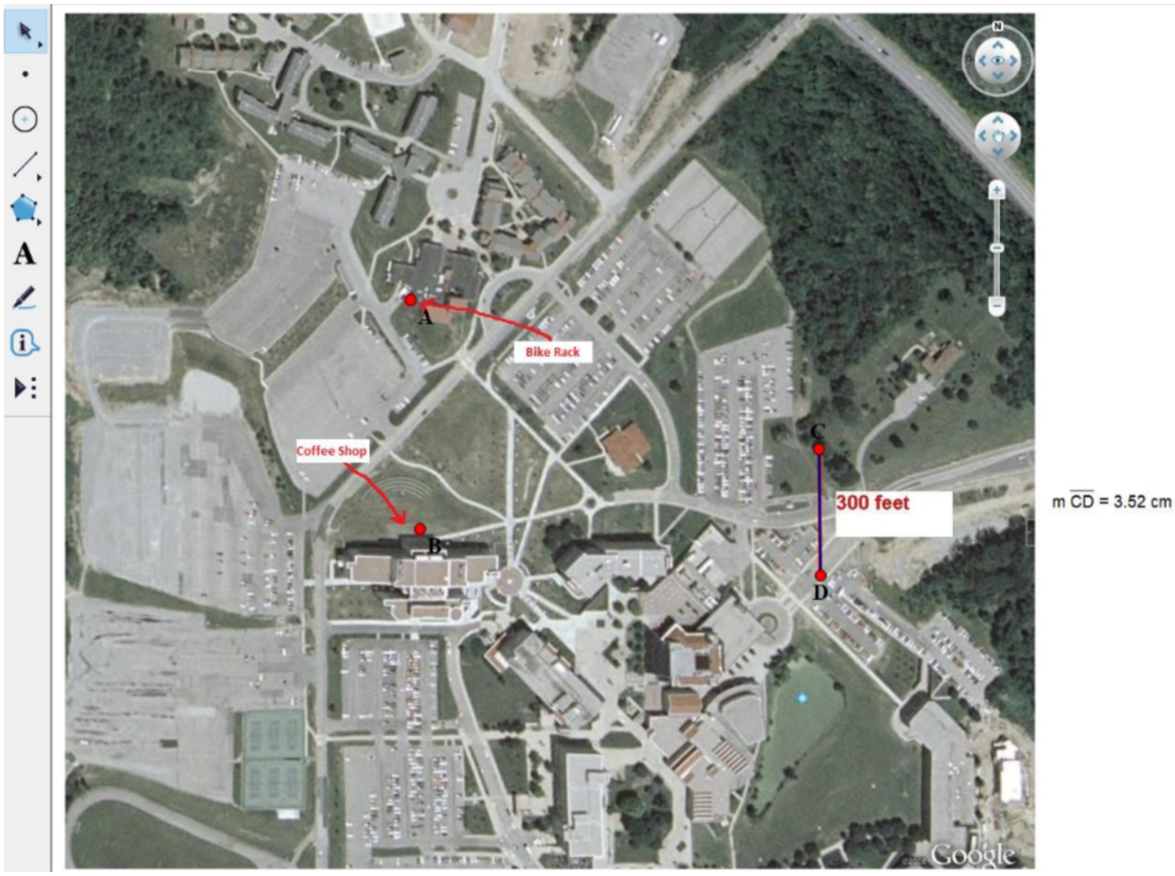
In this case, we are using Geometer's Sketchpad as the dynamic geometry environment. We begin by importing an electronic map into the environment as illustrated in Figure 7.

Figure 7: *Map imported into Geometer's Sketchpad.*



Knowing that there are two fixed points at the coffee shop and bike rack, we can use the Point Tool to place points A and B those locations. Also, we will need a fixed constant, representing the length of $1,596$ feet. Using the given length of 300 feet for reference, we would like to create a line segment representing $1,596$ feet. To do this, we can measure the given length of 300 feet on the map using the Measure Tool, after creating a line segment for that length using the Line Segment Straightedge Tool. In this case, that segment is having length 3.52 cm. This is illustrated in Figure 8.

Figure 8: *Map with points and measured line segment.*



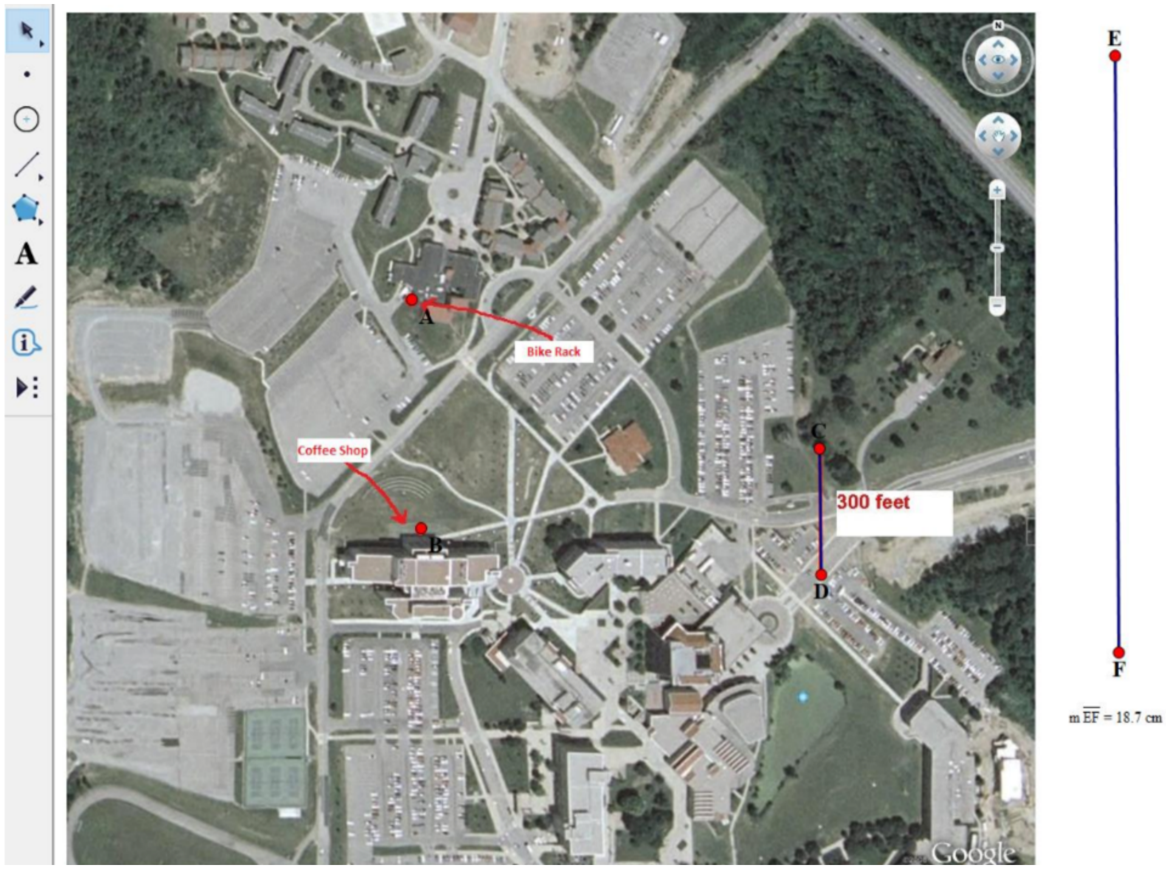
Now, setting up a proportion using the scale factor 300 feet = 3.52 cm, we have

$$\frac{300 \text{ feet}}{3.52 \text{ cm}} = \frac{1,596 \text{ feet}}{d \text{ cm}}$$

Solving this proportion gives $d \approx 18.7$ cm.

So, we need to create a line segment that is 18.7 cm long. We do this by using the Line Segment Straightedge Tool, measuring its length, then extending the segment to the required length. In this case, segment $\overline{EF}$ is created with that length. This is illustrated in Figure 9.

Figure 9: *Map with points and constant length line segment.*



This length will then be the fixed length on the map within Sketchpad representing the sum of the distances from the coffee shop to a loot location and the loot location to the bike rack. Next, we place an arbitrary point G on that fixed line segment and create two line segments: $\overline{EG}$ and $\overline{FG}$. Then, we select point A and segment $\overline{EG}$ and construct a circle with center at A and radius $m(\overline{EG})$. Then, we select point B and segment $\overline{FG}$ and construct a circle with center B and radius $m(\overline{FG})$. Finally, we place points at the intersections of these two circles and trace their intersections. Then, as point G is moved along line segment $\overline{EF}$, the curve of all possible loot locations is traced out on the map as illustrated in Figure 10. This curve is an ellipse with foci at A and B and fixed constant $m(\overline{EF})$.

This additional work using dynamic geometry serves to solidify the understanding of the definition of the ellipse by providing immediate feedback about the shape of the ellipse according to its definition. If desired, students could then be led to analytically derive the equation of an ellipse given its foci and fixed constant on a rectangular coordinate system. While this is a (+) standard in the Common Core Standards for Mathematics meaning, "Additional mathematics that students should learn in order to take advanced courses such as calculus, advanced statistics, or discrete mathematics," (NGA Center & CCSSO, 2010, p. 57) it would certainly be a worthwhile task for stronger students to undertake.

Figure 10: *Map tracing out the ellipse.*

Conclusion

To many students, conic sections can be difficult to fully comprehend geometrically, often leaving the derived formulas sterile and without understanding. This activity is intended for students to develop a conceptual understanding of one conic section—the ellipse—by actualizing it; indeed actually creating it by hand and walking it out!

Finally, and lest the students (or instructor) believe this real-world situation is implausible, one might consider the case of Melissa Brannen, a young girl who went missing in Virginia in December 1989. The wife of the person accused of her abduction, suspecting her husband of foul play, recorded the odometer readings on his truck for the day and investigators mapped out an elliptical region to create a search area for Melissa. This story was told in the first season, eighth episode of the television show FBI Files (Nelson, 1999) and was inspirational for this activity.

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