An Interesting Analysis on a Curious Convergence Series

Pritam Biswas & Ritam Sinha, *Ramakrishna Mission Vivekananda Centenary College—West Bengal, India*

Abstract: *The authors extend the work of Kempner as they investigate a peculiar convergence series derived from the harmonic series through the exclusion of terms containing specific digits in their decimal representation. The authors introduce a novel set S and its corresponding series S', which converge under certain digit-exclusion criteria. By generalizing this approach, they explore the mathematical and pedagogical implications of such series, highlighting their potential to enrich student experiences with proof and inspire further research in series convergence. This work not only broadens the understanding of Kempner's series but also invites educators and mathematicians to reconsider the convergence of series through the lens of digit exclusion.*

Keywords: *Series convergence, digit exclusion, mathematical education*

Introduction

It is well known that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges. However, Kempner (1914) showed that if we exclude those terms for which n contains at least one '9' in its decimal representation, then the series converges. Consider, for instance, assuming that all terms with a '9' in the decimal representation of the denominator are omitted, the following series will converge.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16} + \frac{1}{17} + \frac{1}{18} + \frac{1}{20} + \cdots$$

Remarkably, in the same paper, Kempner showed that when *any* integer in the set 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 is excluded, then the remaining series converges. This concept has been further explored, with various convergent subseries of the harmonic series obtained by excluding digits other than '9', as discussed by Wadhwa (1975), Craven (1965), Irwin (1916), Mukherjee and Sarkar (2021), and Saha, Pal, and Chakraborty (2023).

In this study, we build upon Kempner's foundational work by investigating a series that converges through the exclusion of terms based on a distinctive criterion: the presence of the digit '9' in the decimal places corresponding to even powers of 10 (for instance, ones ($1 = 10^0$), hundreds ($100 = 10^2$), and ten-thousands places ($10000 = 10^4$) places in the number's decimal representation). This methodology gives rise to a novel set S, and its associated series S, shedding new light on the behavior of series under unique digit-based exclusion criteria. This exploration advances our understanding of series convergence and offers a dynamic method for engaging with mathematical concepts. By extending the digit-exclusion technique introduced by Kempner, our work encourages a deeper examination of series convergence and fosters opportunities for further scholarly inquiry and educational practice, thereby enriching the dialogue within the mathematical community and beyond.

Constructing and Analyzing the Digit-Exclusion Series

The series under investigation, denoted as S', is defined based on a specific exclusion criterion and is expressed as:

$$S' = \sum_{n \in S} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\frac{1}{17}+\frac{1}{18}+\frac{1}{20}+\cdots$$

Notice we exclude terms such as $\frac{1}{9}$, $\frac{1}{19}$, $\frac{1}{29}$, $\frac{1}{109}$, $\frac{1}{4923}$ (but not $\frac{1}{92}$ or $\frac{1}{191}$). Curious why? If it's not immediately clear, a careful reread might illuminate the pattern!

Proving Convergence

To understand the convergence of the infinite series S', we split S' into segments, a_i, suggested below.

$$\begin{aligned}
a_1 &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}, \\
a_2 &= \frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\ldots+\frac{1}{98}, \\
a_3 &= \frac{1}{100}+\frac{1}{101}+\ldots+\frac{1}{998}, \\
&\vdots \\
a_n &= \text{segments similarly constructed, with each term adhering to our digit-exclusion criteria.}
\end{aligned}$$

Note that each segment a_i extends up to the next occurrence of the digit '9' in the relevant decimal positions. The series resumes after each segment where terms containing the digit '9' in excluded decimal places are omitted. Each a_i thus spans all permissible terms up to the point where a term would be excluded based on our criterion. The meticulous construction of these segments allows us to approach the convergence of S' from a foundational perspective.

To further establish the convergence of S', we now turn our attention to estimating an upper bound for the sums represented by each segment a_i. This process involves considering the sums of reciprocals of integers within specific ranges, methodically excluding those integers that contain the digit '9' in decimal places corresponding to even powers of 10. By determining these upper bounds, we lay the groundwork for a rigorous proof of convergence, rooted in the properties of geometric series.

Intriguingly, the upper bound for each sum can be represented as a product involving powers of 9 and powers of 10. The power of 9 corresponds to the number of terms that do not contain the digit '9', while the power of 10 reflects the position of the digit '9' being excluded. Specifically, here's how the initial terms a_1 to a_4 are bounded, leading to a pattern that can be extended to any term a_k:

$$\begin{aligned}
a_1 &< 8 \times \frac{1}{1}, \\
a_2 &< 8 \times 9 \times \frac{1}{10}, \\
a_3 &< 8 \times 9^2 \times \frac{1}{10^2}, \\
a_4 &< 8 \times 9^3 \times \frac{1}{10^3},
\end{aligned}$$

and in general, for the k^{th} term, the upper bound can be simplified and consistently applied as follows:

$$a_k < 8 \times 9^{k-1} \times \frac{1}{10^{k-1}}.$$

This formulation implies that each a_i is constrained by a decreasing count of terms (denoted by the factor of 8) and scaled by increasing powers of 9 and 10. The resulting pattern demonstrates clear parallels with a geometric series, where each term diminishes relative to the preceding term by a consistent factor.

Given the upper bound for each segment ai demonstrates a geometric progression with a ratio less than 1, we invoke the Comparison Test to conclude that the series S' converges. This is because each segment's upper bound aligns with the sum of a geometric series, which is known to converge when its common ratio is between -1 and 1.

Generalization and Educational Implications

Following the proof, we extend the discussion to the generalization of this concept to other digits and its implications for mathematical education. This section will delve into how digit-exclusion series can be utilized as a pedagogical tool, offering students a unique lens to explore series convergence and to foster deeper mathematical inquiry.

Generalization of Digit-Exclusion Criteria

In generalizing the digit-exclusion criteria, students can consider the effects of excluding each digit from 0 to 8 in turn, revealing a consistent pattern of convergence across various digit-exclusion series. This generalization underscores the robustness of the digit-exclusion method in studying series convergence, providing a broad foundation for further exploration.

Pedagogical Implications

An initial exploration of digit-exclusion series such as the one presented in this paper can serve as a springboard for discussions on mathematical reasoning and proof techniques, enriching students' mathematical experiences and understanding. For instance, students can use computational tools to simulate digit-exclusion series, allowing them to visualize how altering the exclusion criterion affects convergence. Such an approach can demystify abstract concepts and make the mathematics more accessible. A few approaches include the following:

- **Desmos** offers a powerful graphing calculator that can be used to create interactive math visuals. While it doesn't have a pre-built digit exclusion capabilities, students can use Desmos to graph series and explore their properties.
- **GeoGebra** is a dynamic mathematics software that combines geometry, algebra, statistics, and calculus in one easy-to-use package. Students can create interactive applets with GeoGebra that implement digit exclusion or series convergence.
- **Wolfram Demonstrations Project**: This platform hosts an extensive collection of interactive demonstrations created with Wolfram Mathematica. Demonstrations related to series convergence can be adapted to include digit exclusion criteria. The platform also includes tools for students to develop their own custom demonstrations.
- **Coding Platforms**: For those with programming experience, creating a custom applet using JavaScript and HTML5 or Python (with libraries like Matplotlib for graphing) can be a rewarding project. This approach allows for complete customization, enabling you to define digit exclusion criteria and visualize series convergence.
- **Mathematica or MATLAB**: Both are powerful tools for mathematical computation and visualization. If you students have access to either, they can write a script to compute partial sums of a series under various digit exclusion criteria and plot these sums. These platforms offer extensive documentation and community forums where students might find similar projects or helpful guidance.

Conclusion

This study builds upon Kempner's work by introducing a novel series derived through digit-exclusion criteria, enriching the landscape of mathematical series with fresh insights and implications for both research and education. By broadening the scope of series that exhibit peculiar convergence properties, we invite ongoing inquiry into the nature of convergence and its educational applications, fostering a deeper appreciation for the beauty and complexity of mathematics.

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Pritam Biswas and **Ritam Sinha** are master's students within the Department of Mathematics at Ramakrishna Mission Vivekananda Centenary College.