Gabriel's Horn and the Painter's Paradox in Perspective

Richard Kaufmann
Founder, Office Expander (www.officeexpander.com)

Abstract: *Gabriel's Horn is usually discussed as the painter's paradox. The horn can hold a finite volume of paint, but its inner surface area is infinite and, therefore, cannot be painted. This may seem counterintuitive at first. In this paper, we provide the following perspective: Any finite volume consists of an infinite number of area layers, which amounts to an infinite surface area. This is shown using an example of a "mathematical" ice cube that melts into an infinitely thin film of infinite surface area. Students can appreciate this before they encounter calculus, which is normally used to establish the painter's paradox. So, we show a perspective that is accessible to a wider range of students that is also generalizable to all volumes.*

Keywords: *Painter's Paradox, Gabriel's Horn, perspective, dimension, infinite*

Background

Gabriel's horn can hold a finite volume of paint, but its inner surface area is infinite and, therefore, cannot be painted. Are we able to teach a lesson on Gabriel's Horn to students before they encounter calculus, which is normally used to establish the painter's paradox? In this paper, we will attempt to do so, such that students can develop deeper insights into the relative nature of dimension and concepts (such as summing infinitesimals) that they will later encounter in calculus.

One of the key concepts covered by math teachers is the idea of dimension. Traditionally, the concept of point, which has no breadth, precedes the concept of finite length, which consists of an infinite number of points. That is, students are able to grasp that an infinite number of points from one dimension forms a finite length in the next higher dimension. As we increase dimensions, an infinite number of lines can form a finite plane, and then an infinite number of finite planes can form a 3-D object, such as a cube. Each new dimension requires an infinite number from the lower dimension.

Because students can appreciate this they can learn some new and interesting concepts before they encounter calculus. Imagine we ask students the following question. Is it possible to have a cube or another 3-D shape which is filled to the brim with paint but for which that amount of paint would not be enough to paint the inside? Students may come up with answers, including that this does not seem possible. They may state that, since a 3-D cube has enough paint to cover the volume, then certainly it has enough paint to cover the inner surface. They may be surprised to learn that there is just such an example where the inside could not be painted, and this is shown by Gabriel's Horn.

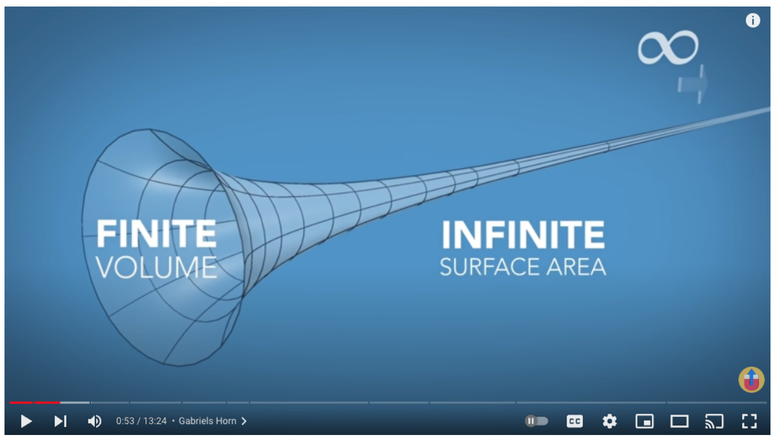
Students may puzzle about an object that can hold an amount of paint which is insufficient to paint the inner surface. After students try to guess how this is possible, we suggest that teachers use technology in order to provide a visual of the problem. YouTube offers short videos that discuss the painter's paradox and Gabriel's Horn (discovermaths, 2019; Epic Math Time, 2019; Learning Curve, 2020; Müller, 2022; Numberphile, 2021; Professor Peter, 2020; Up and Atom, 2021). Video demonstrations can help reinforce classroom discussions and brings the paradox to life. Specifically, we will focus on a video by Up and Atom called, "This Object Has Infinite Surface Area, but Finite Volume," narrated by Jade Tan-Holmes (Up and Atom, 2021).

Gabriel's Horn and the Painter's Paradox

Gabriel's Horn was discovered by Evangelista Torricelli in 1641, as discussed by Coll and Harrison (2014). Their paper uses calculus to show that Gabriel's Horn has a finite volume of π volume units and an (inner or outer) surface area that is infinite. In other words, the horn can be filled with a finite π volume of paint, but its inner surface is infinite and, therefore, cannot be painted. This has become known as the painter's paradox. We believe that students can learn about this before encountering calculus.

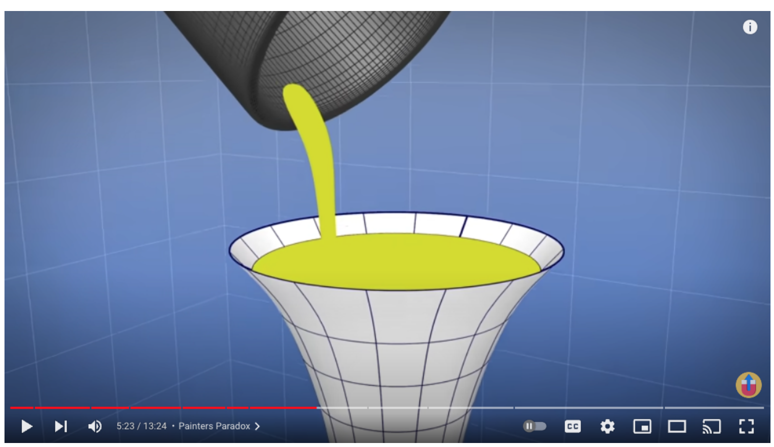
Teachers could highlight the following from Up and Atom's videos. Figure 1 shows Gabriel's Horn with a finite volume and an infinite surface area (on the inside or outside surface of the horn).

Figure 1: *Gabriel's Horn as shown in a YouTube video by Up and Atom (2021)*



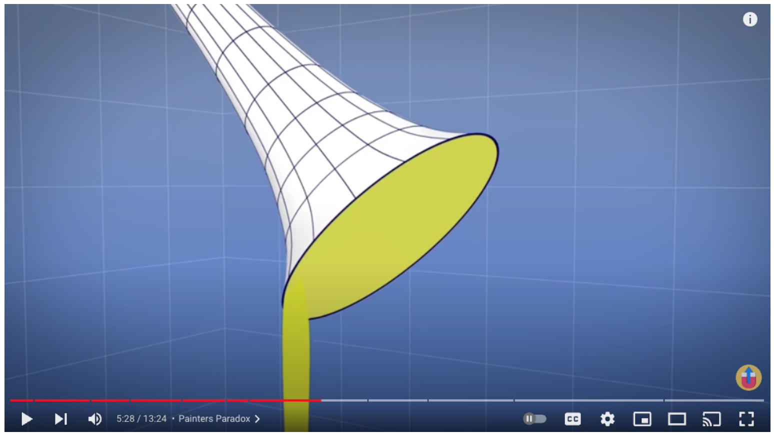
Teachers can show Gabriel's Horn filled with a finite amount of paint in Figure 2.

Figure 2: *Gabriel's Horn filled with a finite amount of paint.*



Finally, teachers can show Gabriel's Horn emptied of paint in Figure 3. Here it appears that the inner surface area of the horn is "painted" due to the filling and subsequent emptying of paint. However, since the surface area is infinite, then it could not be expected that a painter could actually paint the infinite surface area: the "painter's paradox."

Figure 3: *Gabriel's Horn emptied of paint, with the inner surface area "painted."*



The horn initially challenged mathematicians, especially before calculus rigorously established the result.

Previous Contexts for the Painter's Paradox

In the present paper, we do not rederive the calculus used for the problem. Rather, we elaborate and highlight insights into the problem, such as those stated or inferred by Coll and Harrison and others. We also attempt to make these insights accessible to a wider audience of students without requiring exposure to calculus. Teachers can tailor the level of discussion to their students. Later, we provide an example to capture these insights.

Coll and Harrison point out (2014):

> The paradox . . . disappears when we realize that it is the difference in dimensions that creates the illusion of impossibility. We cannot paint the two-dimensional Horn with a uniformly thick coating of three-dimensional paint. For the Horn, we would have to reduce the thickness as we paint . . . If we are careful, and reduce the thickness quickly enough, any amount of paint will do: We could use an eyedropper full of paint to coat the can.

Their paper, along with other sources mentioned, appears to highlight the following insights into the painter's paradox:

1. There is a difference between the dimension of area and the dimension of volume.
2. Similar to the previous point, the numerical quantity of area cannot be compared to the numerical quantity of volume, since this would be an error in dimensional analysis (i.e., $8cm^2 \not> 4cm^3$, since the units are not consistent).
3. Real paint, consisting of atoms, would eventually not fit into the ever-decreasing cross-sectional area of the horn. A distinction is made between real paint and "mathematical" paint, which would not consist of atoms, but rather a continuous fluid that could flow throughout the horn.
4. The painter's paradox is not really a paradox at all. It is a valid mathematical derivation. The horn is an abstraction for an infinite object, rather than a physical object.

Again, we focus on the video by Up and Atom (2021) because they really try to get at the heart of the painter's paradox. Even after discussing dimensional analysis, etc., they state (starting at 8:03):

> Mathematically, there is no paradox with a shape having infinite surface area and finite volume. There are other shapes that are infinite in one dimension and finite in another, like the Koch snowflake, which may be easier to visualize. It's a fracture which follows

a simple iterative process. You end up with an infinitely long perimeter, but a finite area. These objects exist in the mathematical world. And when I say exist, I mean that they're consistent with the laws of that world. So that was the technical resolution to the paradox, that there is in fact no paradox. By trying to apply paint to an abstract volume and surface area, we [were] inadvertently comparing two incompatible quantities. *Basically, don't try and paint infinite mathematical objects. But what if we did?* [emphasis added]. I feel like this video wouldn't be complete unless we at least talked about what might happen.

Some interpretations are then discussed, including the following quotes from the video:

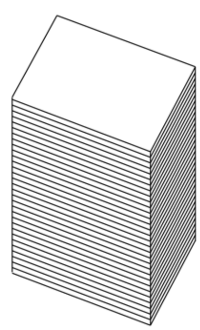
- Physical paint has volume and thickness.
- Paint doesn't have to have thickness.
- Some people might say that it's a waste of time to even think about these kinds of questions.
- . . . even simple ideas like area and volume are more complicated than we might think.
- To have a purely mathematical object that can't exist in the real world raises some interesting questions. Does math exist only in our minds? Is it something we've invented to make sense of the world around us? Or does it exist outside of us and we discover it?

In the next section, we elaborate on what has previously been discussed about the painter's paradox. We highlight a perspective that could be insightful to students. Our aim here is to provide an alternative understanding of the problem and give an explanation that a wider audience could understand.

Example and Perspective

A key point in resolving the "paradox" can be seen by imagining a cube of finite volume that sits on an infinitely long table. The cube consists of an infinite stack of squares with zero thickness, as shown in Figure 4. Teachers might use a stack of papers to illustrate this (and then imagine spreading the papers on a table and imagining the paper thickness getting smaller and smaller as the number of pages increases to infinity).

Figure 4: *Cube consisting of an infinite stack of squares with zero thickness: shown with exaggerated separation between squares.*



Since there are an infinite number of squares, then we can tile the infinite table with squares in any way that we like. As long as the squares do not lay on top of each other, the combination of squares will always encompass an infinite area on the table. This works no matter the size of the cube.

We could also imagine this cube to be an ice cube that melts into an infinitely thin film with an infinite surface area. This example of a "mathematical ice cube" could not be made of atoms or form a film of water with any thickness.

We consider this insight as the key perspective into the painter's paradox. Any 3-dimensional volume consists of an infinite number of 2-dimensional shapes stacked together. Even a small piece of a

3-dimensional volume consists of an infinite number of 2-dimensional shapes. The combined surface area of these infinite 2-dimensional shapes is always infinite.

And this is the case for Gabriel's Horn and the painter's paradox. Even a small volume of the paint shown inside the horn in Figure 2 could be thinned out into an infinite area that covers the entire horn's surface area.

And no calculus is required for students to see this result.

Final Remarks

Gabriel's Horn may have been one of the first times that seventeenth-century mathematicians really tried to grapple with a 3-dimensional volume requiring an infinite 2-dimensional surface area. The key perspective from this paper shows that a 3-dimensional volume is always made of an infinite number of 2-dimensional shapes. This applies to a cube, Gabriel's Horn, or any volume- so we see that there is no paradox. Pre-calculus students can comprehend all of this through just an understanding of the relative nature of dimensions.

References

Coll, V. E., & Harrison, M. R. (2014). Gabriel's Horn: A Revolutionary Tale. *Mathematics Magazine, 87*(4), 263–275. https://doi.org/10.4169/math.mag.87.4.263

discovermaths. (2019, July 31). Gabriel's Horn and the painter's paradox [Video]. *YouTube*. https://www.youtube.com/watch?v=61Ofa0GlUUU

Epic Math Time. (2019, March 18). Gabriel's Horn and the Painter's Paradox [Video]. *YouTube*. https://www.youtube.com/watch?v=iubH2WmpPLw

Learning Curve. (2020, April 5). The Painter's paradox or Gabriel's horn paradox [Video]. *YouTube*. https://www.youtube.com/watch?v=Ex8Pg_bGWGM

Müller, K. (2022, September 1). The Painter's Paradox and the Mystery of Gabriel's Horn. *Medium*. https://www.cantorsparadise.com/the-painters-paradox-and-the-%20mystery-of-gabriel-s-horn-41f668b56559

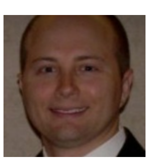
Numberphile. (2021, February 18). Gabriel's Horn Paradox [Video]. *YouTube*. https://www.youtube.com/watch?v=yZOi9HH5ueU

Professor Peter. (2020, June 23). Torricelli's Trumpet (or Gabriel's Horn): A Paradox of Area and Volume [Video]. *YouTube*. https://www.youtube.com/watch?v=mXfFWQ_x6RE

Up and Atom. (2021, September 24). This Object has Infinite Surface Area, but Finite Volume [Video]. *YouTube*. https://www.youtube.com/watch?v=3WVpOXUXNXQ

Acknowledgements

The author thanks Kara Kaufman for editing this paper. The author also wishes to thank the anonymous reviewers for their suggestions on improving this paper and Jade Tan-Holmes for permission to use the images in Figures 1-3.



Richard Kaufman (rdkaufman01@gmail.com) received his B.S. in Information Technology in 2014, his M.S. in Applied Math in 2011, and his M.S. in Mechanical Engineering in 2000, all from UMass Lowell, USA. Richard is the founder of Office Expander (www.OfficeExpander.com) and is a registered Professional Engineer.