Delving Deeper: Squares within Squares and Cubes within Cubes

Ruby Schwan & Sarah Daye

Miami University

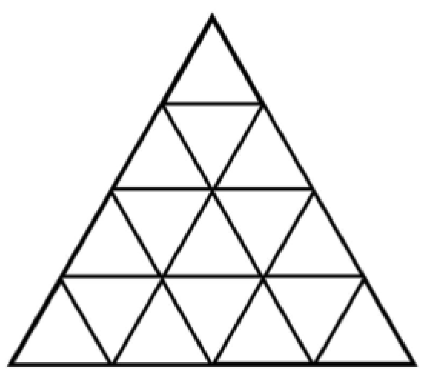
Abstract: *This article examines the steps to find out how many squares can be drawn within a 100 by 100 array of points. Strategies of pattern analysis and organization are discussed, and the problem is then extended to include a spatial awareness component. The writers solve these fascinating problems step-by-step and then explain how they could be implemented in the high school Geometry classroom.*

Keywords: *Problem Solving, Patterns, Sense-Making*

Introduction

Have you ever encountered an instance where you were asked to determine how many triangles appear to be in an array of multiple triangles? Sometimes these instances occur online as a brainteaser or even in your high school geometry classroom. The answer to these types of problems can be quite surprising. Most students or individuals count the obvious triangles but don't consider all possibilities. Let's consider the array of triangles in Figure 1.

Figure 1: *How many triangles do you see?*



Many see 16 triangles. But the answer is more complex than that. For starters, yes there are 16 small individual triangles (unit triangles) that make up one large triangle ($16 + 1 = 17$ triangles). Next, we want to look beyond the individual unit triangles and determine what other size triangle we can make (3 unit triangles on the base with one unit triangle on top, 5 unit triangles on the base with 3 unit triangles in the middle and 1 unit triangle on top). All of the possible sizes of triangles and the number of triangles created from that size are found in Table 1.

Table 1: *Type and number of triangles within Figure 1.*

Size 1	Size 2	Size 3	Size 4
			
16 (size 1) triangles	7 (size 2) triangles	3 (size 3) triangles	1 (size 4) triangle

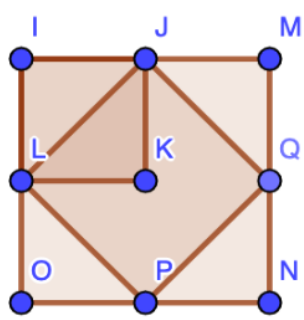
If we sum all of the possible triangles, we get a surprising answer of 27 unique triangles that are made from the array of 16-unit triangles. This answer is much different than what most individuals would have predicted that answer to be.

This brain teaser problem is a great discovery question for high school students enrolled in any geometry class. The Ohio standard G.MG.3 (2017, p. 83) wants students to apply geometric methods to solve design problems. This problem does just that, students can solve this problem by creating organized lists and tables to discover patterns and generalize their findings on a broader scope. Furthermore, it requires spatial problem-solving, which is frequently underemphasized in the high school mathematics curriculum. The question is, can we apply these problem-solving ideas to other geometric shapes?

The Square Problem

The answer is yes! Now that we understand triangles within triangles, let's apply a similar pattern of thinking to understand squares within squares. Suppose you have a 100 by 100 array of points arranged evenly in a square, containing a total of $10,000$ points. How many total squares (T) can be drawn in that array? This total includes every unique square of each dimension (S), which is identified in terms of its area (A). Note that this problem was originally presented by Frederick Stevenson in his essay "Exploratory Problems in Mathematics" (1997, p.87–88). Figure 2 illustrates what a 3×3 array looks like as well as each size square that can be drawn within it.

Figure 2: *Square sizes in a 3×3 array.*



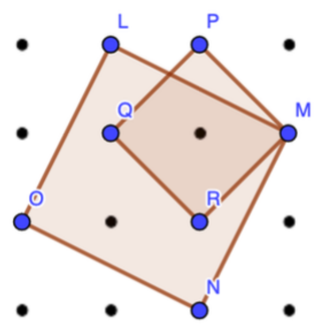
Note that while we only drew 3 total squares in this array, there are actually a total (T) of 6 unique squares that exist in the array because there are 4 1×1 unit squares. This information is formalized in Table 2.

Table 2: *Number of squares in a 3×3 array.*

Array Size: 3x3			
area of square (A)	1	4	2
number of squares (S)	4	1	1

If you have already tried to think through this problem on your own, you might be wondering how we found the square of area 2 in this array. Like the triangle problem from the introduction, this problem is more complicated than it seems. Not only do we have to count the squares whose sides are vertical and horizontal, but we also have to count what we call "diagonal squares." Square $JLPQ$ in Figure 2 is the only diagonal square in a 3×3 array and its area can easily be shown to be 2 under the assumption that the distance between 2 points is 1 unit. Figure 3 shows another possible diagonal square in a 4×4 array with an area of 5.

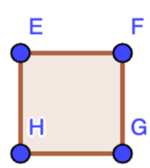
Figure 3: *Oblique squares in a 4×4 array.*



Exploration of the Square Problem

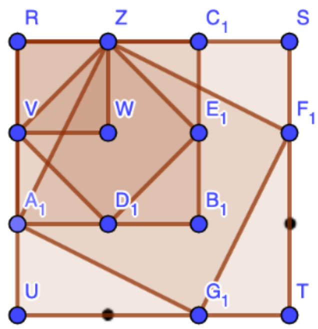
Before we can formulate the tables for each size array, like Table 2, we must explore each case by drawing squares. Using Square Dot Paper in Geogebra, we were able to draw each array and record the number of squares for arrays of size 2×2 through 7×7. First, consider the 2×2 case. Figure 4 shows that only 1 square can be formed by this set of 4 points.

Figure 4: *Square in a 2×2 array.*



We have already examined the 3×3 case, so let's now consider the case of the 4×4 array. Figure 5 shows the 5 different square sizes that can be formed in this array. It is important to note that the number of square sizes increases as the size of the array increases.

Figure 5: *Square sizes in a* 4×4 *array.*



We continued this square construction process through the 7×7 array and organized our findings in Table 3.

Table 3: *Number of squares in arrays of various dimensions.*

Array Size: 2x2	
A	1
S	1

Array Size: 3x3			
A	1	4	2
S	4	1	1

Array Size: 4x4					
A	1	2	4	5	9
S	9	4	4	2	1

Array Size: 6x6											
A	1	2	4	5	8	9	10	13	16	17	25
S	25	16	16	18	4	9	8	2	4	2	1

Array Size: 7x7															
A	1	2	4	5	8	9	10	13	16	17	18	20	25	26	36
S	36	25	25	32	8	16	18	8	9	8	1	2	4	2	1

Finding the Formula

Now let's consolidate the information in Table 3 by recording the total number of unique squares (T) per array size. Table 4 displays this information.

Table 4: *First through fourth order differences in total number of squares (T) per array.*

Array Size	NxN	T	1st Difference	2nd Difference	3rd Difference	4th Difference
2x2	4	1				
3x3	9	6	5			
4x4	16	20	14	9		
5x5	25	50	30	16	7	
6x6	36	105	55	25	9	2
7x7	49	196	91	36	11	2

Unfortunately, there is no clear pattern to discern just by examining the first three columns of Table 4, as we would hope. We decided to employ the "Difference Method," seen in the last four columns to determine if our formula could be represented as a polynomial. The first difference represents the difference in T values for consecutive rows of arrays. As you might guess, the second row represents the difference in the first differences for consecutive rows, and the pattern continues for the third and fourth differences. The method stops when all consecutive pairs of rows yield the same difference; since our 4th difference is 2 between the 5×5 and 6×6 arrays and between the 6×6 and 7×7 arrays, we know that the T formula we are searching for is quartic. In other words, our final formula should be of the form,

$$T = AN^4 + BN^3 + CN^2 + DN + E, \text{ with } A, B, C, D, E \in \mathbb{R}$$

Since we have 5 unknown variables in this formula, we need to create 5 equations to solve for them. We will use the T and N values for the 2×2 through the 6×6 arrays to obtain the following system:

$$1 = A \cdot 4^4 + B \cdot 4^3 + C \cdot 4^2 + D \cdot 4 + E$$
$$6 = A \cdot 9^4 + B \cdot 9^3 + C \cdot 9^2 + D \cdot 9 + E$$
$$20 = A \cdot 16^4 + B \cdot 16^3 + C \cdot 16^2 + D \cdot 16 + E$$
$$50 = A \cdot 25^4 + B \cdot 25^3 + C \cdot 25^2 + D \cdot 25 + E$$
$$105 = A \cdot 36^4 + B \cdot 36^3 + C \cdot 36^2 + D \cdot 36 + E$$

With a little help from a powerful calculator, we can obtain the solution to this system and substitute the values in to obtain the formula:

$$T = \frac{1}{12}N^4 - \frac{1}{12}N^2$$

Now, we can use this formula to solve our original problem. Recall that we have a 100×100 array, so we substitute $N = 100$ into our formula to find that the total will be 8,332,500 squares.

$$T = \frac{1}{12} \cdot 100^4 - \frac{1}{12} \cdot 100^2 = 8,332,500$$

Extension: The Cube Problem

Suppose we extend this idea and have a 100 by 100 by 100 array of points arranged evenly in a cube, containing a total of $1,000,000$ points. How many total cubes of a particular size (C) can be drawn in this array? This total includes every unique cube of each dimension, which is identified in terms of volume (V). Note that this problem was also obtained from Stevenson's "Exploratory Problems in Mathematics" (1997, p.87–88). Table 5 illustrates what this means using a $3 \times 3 \times 3$ array as an example.

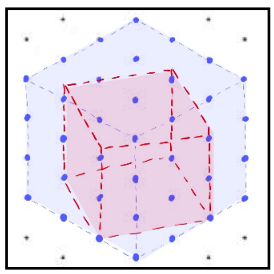
Table 5: *Number of cubes in a* $3 \times 3 \times 3$ *array.*

Array Size: 3x3x3		
size of cube (V)	1	8
number of cubes (C)	8	1

Therefore, in a $3 \times 3 \times 3$ array, there are two distinct-sized cubes, 8 cubes with volume 1 and 1 cube with volume 8 cubic units.

Before we begin exploring the cube problem, it is important to note that there will not be any diagonal cubes like there were diagonal squares in the square problem. The reason for this is that the side length of the cube would not be a whole number, thus not meeting all of the points in the array on the vertices of the cube. Figure 6 shows a 4×4 array of points and how the height of the cube fails to meet a point.

Figure 6: *A cube with side length* $\sqrt{5}$.

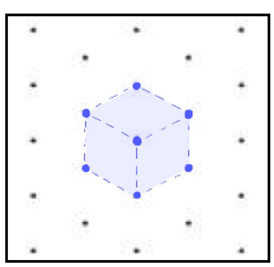


Exploration of the Cube Problem

Similar to the exploration of the square problem, we can generate tables for different-sized arrays. Before generating the tables, it would be beneficial to explore all cases by drawing the arrays on isometric paper and recording the number of cubes.

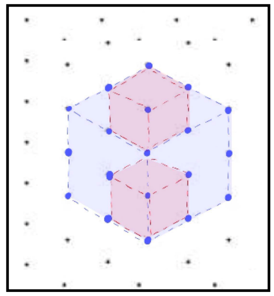
First, consider the $2 \times 2 \times 2$ array of points. Figure 7 shows that there is only one cube with a volume equal to 1 cubic unit.

Figure 7: $2 \times 2 \times 2$ *array with 1 cube.*



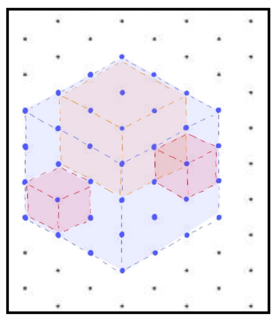
Next we can explore a $3 \times 3 \times 3$ array of points. In this array, there will be two distinct-sized cubes that we can make. Figure 8 shows there will be 8 cubes with $V = 1$ and 1 cube with $V = 8$. Therefore an array of $3 \times 3 \times 3$ points has 9 total cubes.

Figure 8: $3 \times 3 \times 3$ *array with 9 cubes.*



Finally, consider a $4 \times 4 \times 4$ array of points. In this array, there will be three distinct-sized cubes that we can make. Figure 9 shows there will be 27 cubes with $V = 1$, 8 cubes with $V = 8$, and 1 cube with $V = 27$.

Figure 9: $4 \times 4 \times 4$ *array with 36 cubes.*



Similarly to the square problem, we can generate tables for different arrays starting with $2 \times 2 \times 2$ and ending with $7 \times 7 \times 7$.

Table 6: *Number of cubes in arrays of various dimensions.*

Array size: 2x2x2	
V	1
C	1

Array size: 3x3x3		
V	1	8
C	8	1

Array Size: 4x4x4			
V	1	8	27
C	27	8	1

Array Size: 5x5x5				
V	1	8	27	64
C	64	27	8	1

Array Size: 6x6x6					
V	1	8	27	64	125
C	125	64	27	8	1

Array Size: 7x7x7						
V	1	8	27	64	125	216
C	216	125	64	27	8	1

Finding The Formula

Just as we found the formula to compute the number of total squares in an $N \times N$ array of points, we can use the same method to derive a formula for calculating the total number of cubes (T) in an $N \times N \times N$ array of points.

Table 7: *First through fourth order differences in total number of cubes (T) per array.*

Array Size	NxNxN	T	1st Difference	2nd Difference	3rd Difference	4th Difference
2x2x2	8	1				
3x3x3	27	9	8			
4x4x4	64	36	27	19		
5x5x5	125	100	64	37	18	
6x6x6	216	225	125	61	24	6
7x7x7	343	441	216	91	30	6

Unlike the square problem, there is a noticeable pattern present in the first three columns. Specifically in the third column labeled T (total number of cubes), each number is a square number. But the square numbers are not consecutive square numbers; therefore, we need to discover other hidden patterns. We used the "Difference Method" in the same manner as the square problem. We found the first difference of the rows in T, then the second difference in the rows of the first differences, and so on until we found a constant difference occurring in the fourth difference column. Once again, the constant fourth difference informs us that the formula will once again be a quartic equation, so the equation will be of the form,

$$T = AN^4 + BN^3 + CN^2 + DN + E, \text{ with } A, B, C, D, E \in \mathbb{R}$$

We have 5 unknown variables in this formula, and we need to create 5 equations to solve for them. We will use T and N values for the $2 \times 2 \times 2$ through the $6 \times 6 \times 6$ arrays to obtain the following system:

$$\begin{aligned} 1 &= A \cdot 8^4 + B \cdot 8^3 + C \cdot 8^2 + D \cdot 8 + E \\ 9 &= A \cdot 27^4 + B \cdot 27^3 + C \cdot 27^2 + D \cdot 27 + E \\ 36 &= A \cdot 64^4 + B \cdot 64^3 + C \cdot 64^2 + D \cdot 64 + E \\ 100 &= A \cdot 125^4 + B \cdot 125^3 + C \cdot 125^2 + D \cdot 125 + E \\ 225 &= A \cdot 216^4 + B \cdot 216^3 + C \cdot 216^2 + D \cdot 216 + E \end{aligned}$$

Using a powerful calculator to solve for the 5 variables, we can obtain the solution to this system and substitute the values in to obtain the formula:

$$T = \frac{1}{4}N^4 - \frac{1}{2}N^3 + \frac{1}{4}N^2$$

Finally, we can use this formula to solve the original cube problem. Recall that we wanted to know how many cubes are in a $100 \times 100 \times 100$ array of points. We can substitute $N = 100$ into the formula to get a total of $24,502,500$ cubes. Wow!!

$$T = \frac{1}{4} \cdot 100^4 - \frac{1}{2} \cdot 100^3 + \frac{1}{4} \cdot 100^2 = 24,502,500$$

Extend the Extension!

If you have used these tasks in your classroom and want to explore further, consider this extension question:

> How many different sizes of squares (i) can be drawn within different-sized arrays? What about different-sized cubes?

Table 8 includes our preliminary work on the square portion of this problem to get you started.

Table 8: *Tracking i, the number of unique square sizes in the array.*

Array Size	i	Difference in i
2x2	1	
3x3	3	2
4x4	5	2
5x5	8	3
6x6	11	3
7x7	15	4

Summary

The square and cube problem truly resonates as a rich discovery task for students. The combination of creativity, visualizations, and organization enables students to solve higher-level thinking problems such as the square and cube problems presented. The creative aspect of these problems requires students to tap into their artistic sides and discover the possibilities visually. The visualization aspect comes into play when students may need help conceptualizing larger arrays, tapping into their spatial awareness. Finally, students must keep their work organized in some way in order to notice patterns that will lead them to the solution.

These problems would be great discovery tasks to incorporate into a Geometry classroom. As you see, the process of solving a discovery task can be challenging and likely requires multiple work days to complete, but the final result is satisfying and worth the work. This problem truly brings out the mathematicians in students. Students will feel accomplished discovering their very own formulas that answer the initial problems.

References

Ohio Department of Education. (2017). *Ohio Learning Standards for Mathematics*. Author.

Stevenson, F. W. (1997). 25. Squares within Squares and Cubes within Cubes. In *Exploratory Problems in Mathematics* (pp. 87–88). Reston, VA: National Council of Teachers of Mathematics.

Ruby Schwan & Sarah Daye are teacher candidates enrolled in the Adolescent & Young Adult Integrated Mathematics program at Miami University.