Modeling the Functions of Atmospheric Carbon

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Abstract: *For thousands of years, there was a gradual increase of carbon in Earth's atmosphere. Students can model the historical data using linear functions. Then they can learn about climate change in the modern era by, depending on their grade level, using a combination of exponential and periodic functions to explore carbon's multi-faceted variation, distinguishing the recent trend (exponential) from the historical trend (linear) and from seasonal variation (trigonometric).*

Keywords: *mathematical modeling, linear functions, regression*

Modeling the Functions of Atmospheric Carbon

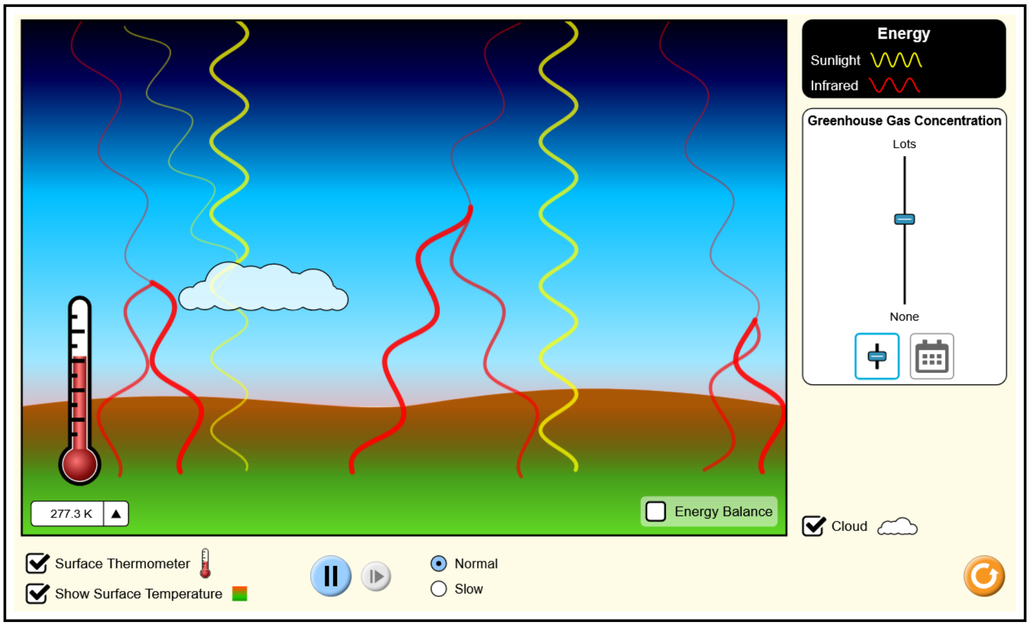
How can mathematics educators help in the effort to slow global warming? When professionals model the trajectory of climate change, they use a great deal of complex mathematics (Neelin, 2010; Teacher's Climate Guide, 2021), but even secondary-level mathematics is sufficient for building crucial understandings of climate change phenomena. In this article, we present a series of three tasks that use linear, exponential, and trigonometric functions to model the Earth's changing amount of carbon in the atmosphere. Our goal is for students to actively engage in aspects of mathematical modeling (Standard for Mathematical Practice 4 [SMP4] in the Common Core State Standards [CCSS], 2010, and the Ohio Learning Standards [OLS]), that is, using mathematical concepts and tools to analyze and understand empirical situations that arise in everyday life. These specific modeling opportunities will allow them to draw on (and enhance) their knowledge of linear versus exponential growth.

Additionally, we anticipate that by engaging in these tasks students will realize the modern changes in atmospheric carbon are not "natural" in the way that previous changes were. In other words, secondary-level mathematical concepts can shed direct light on the global warming issue. Moreover, students will have opportunities to decide if a model is consistent with data (CCSS: S-IC.2, OLS: S.IC.2) and, if they proceed through the full series of tasks, they can also model periodic phenomena (CCSS: F-TF.5, OLS: F.TF.5) and combine multiple functions to represent multi-faceted situations (CCSS: F-BF.1b, OLS: F.BF.1b).

The Situation: Atmospheric Carbon and Climate

For these tasks, we approach the complex issue of global warming by focusing on the simpler phenomenon of carbon in Earth's atmosphere. Atmospheric carbon is typically measured in parts per million, meaning the number of carbon molecules for every million molecules of air. Atmospheric carbon is important because carbon dioxide and other carbon molecules (methane, carbon monoxide, etc.) are the most prevalent greenhouse gases in our atmosphere. Greenhouse gases are molecules that trap infrared photons (aka, heat) on the Earth instead of releasing them out into space. To learn more about the greenhouse effect readers may refer to Shepardson and Hirsch (2019) or explore PhET Interactive Simulations (2022) that demonstrate greenhouse gases and their effect on surface temperatures (Figure 1).

Figure 1: *A screenshot from PhET at the University of Colorado showing infrared energy (red curves) either trapped in the Earth's atmosphere or escaping out to space; a key idea is that increasing greenhouse gases increases the surface temperature.*



Because the level of atmospheric carbon is important with respect to global warming, and because it has been changing dramatically in recent centuries, we can use mathematical functions to represent those changes over time.

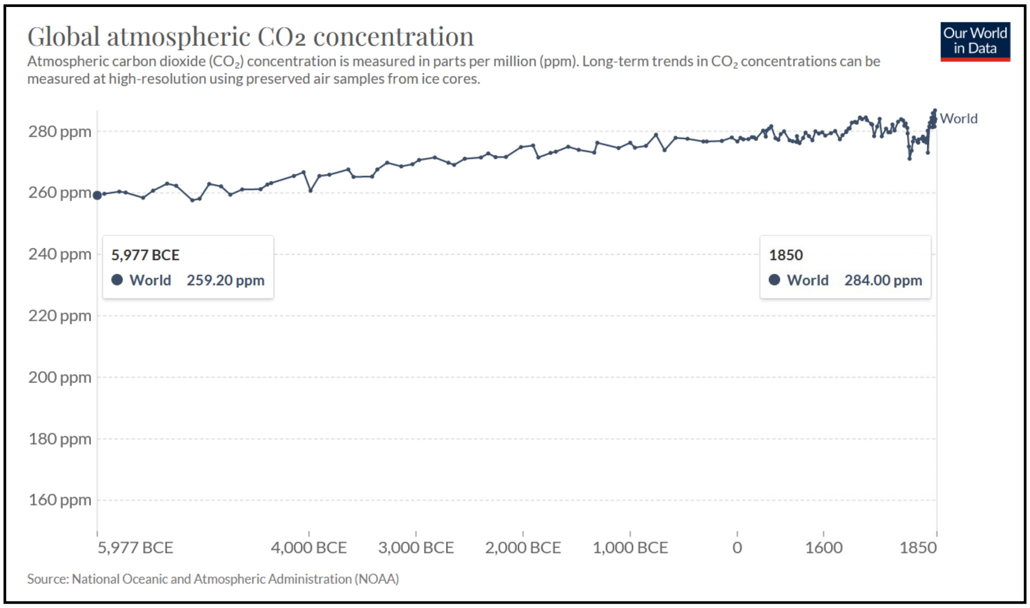
Model 1: Linear Growth of Atmospheric Carbon

Task Description

Using data from the National Oceanic and Atmospheric Administration (NOAA), Figure 2 shows the amount of carbon dioxide in the atmosphere for the years 6000 BCE (Before the Common Era) to 1850 CE (Common Era). Teachers can recreate the graph in Figure 2 by choosing the "long-run series" from NOAA's website and dragging the time endpoints as desired; graphs and tables are both available.

For the first task, teachers can ask students to fit reasonable linear models to these data and then compare students' modeling processes and their resulting lines. Students are welcome to produce slightly different linear models from one another but, based on the actual data, all should have a gradual upward slope.

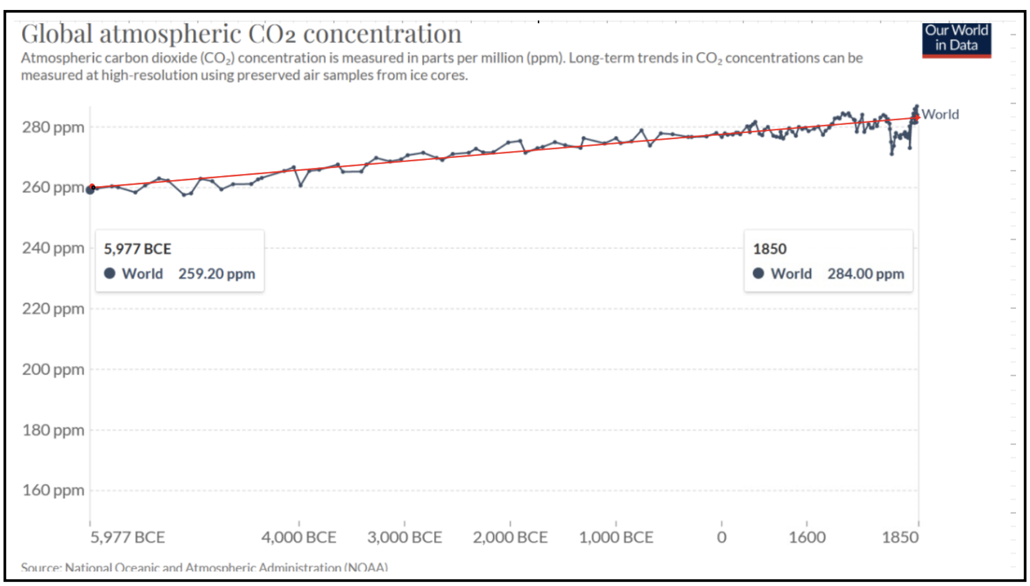
Figure 2: *The amount of carbon dioxide in Earth's atmosphere has gradually increased over most of the past 8000 years. Source: NOAA*



Possible Student Thinking and Discussions

Linear models can be created graphically, with tables, or with equations. When we have implemented this task, we have tended to favor a graphical approach and Figure 3 represents one possible graphical model. Lines such as this can be drawn onto the graph by students; this particular one corresponds to the equation $y = 0.00395x + 276.7$. Students should be invited to interpret terms in the equation in context (0.00395 ppm is the average amount of carbon added to the atmosphere each year; 276.7 ppm is the approximate amount in the year 0 CE).

Figure 3: *A linear approximation of the amount of carbon dioxide in Earth's atmosphere, with the actual data sourced from NOAA.*



Teachers may lead discussions of how the students generated their linear models. In implementing this task with students, we found that some students pick two points as the basis for their line, in which case we might ask, "how did you check that the two points reasonably represented the data?" It might be worth showing two poorly-selected points, such as two that are near one another and result in much too steep a slope. When students produce different but similar lines of fit, we point out that, in real-life scenarios, there can be multiple reasonable models. This also helps to draw out the conversation about the assumptions we make when engaging in mathematical modeling. However, our students still noted the commonalities, such as that "the amount of carbon is growing over thousands of years" and the growth is "linear," with "a little bit more ppm of carbon every year." Furthermore, although a valid linear model can be made within one representation (graph, table, equation), it is worthwhile to compare across the representations by, for example, asking how the slope or y-intercept from an equation show up in the graph or table.

Perhaps most important, students can be asked to extend the task (leading to the next modeling opportunity) by using their linear model to predict the amount of carbon dioxide in the atmosphere in more recent times, such as 1900, 1950, 2000, and 2020, assuming the general linear trend had continued (see Table 1). Teachers can then provide modern data (or have students look them up) so that students may contrast their predictions with the actual measures of atmospheric carbon. Table 1 contains possible predictions as well as the actual measures of atmospheric carbon and it should become clear that the historical model is not working well after 1850. This presents an opportunity to not only make connections to social studies (e.g., what worldwide phenomena started in roughly 1850 and involved taking millions of years' worth of carbon stored beneath the Earth's surface and burning it into the atmosphere? OLS for Social Studies: MWH.8) but also to revise the model given new data, a critical element of mathematical modeling (CCSS: SMP 4).

Table 1: *Predictions of atmospheric carbon after 1850 and the actual measurements according to the NOAA/ESRL global monitoring division.*

Year	Model Predictions (ppm of carbon)	Actual Measures (ppm of carbon)	Error
1900	284.2	294.2	3.5%
1950	284.4	312.8	10.0%
2000	284.6	369.6	29.9%
2020	285.0	412.5	44.8%

Model 2: Exponential Growth of Atmospheric Carbon

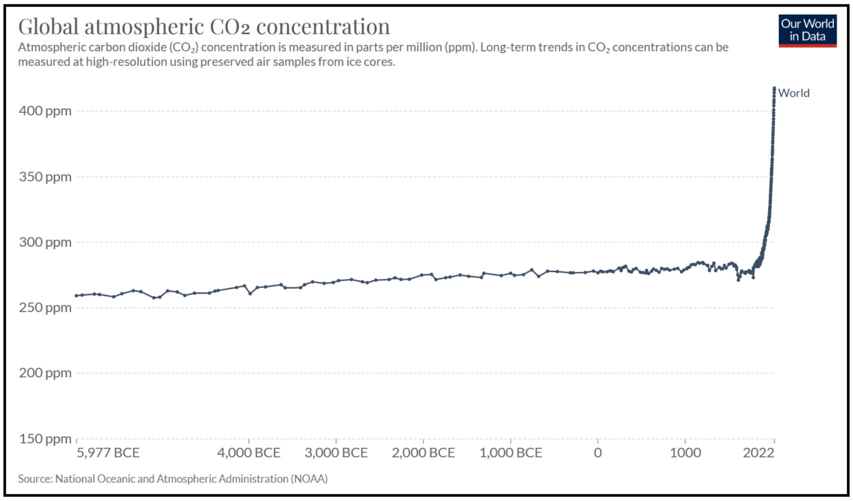
Task Description

The linear model that worked well for data prior to 1850 fails in modern times, as shown by the increasing errors in Table 1. To emphasize this fact, students can explicitly consider whether the recent data is consistent with the historical linear model. A sample writing prompt follows:

> Suppose someone online wrote the following about the atmospheric carbon trend from 8000 BCE – 1850 CE and the more recent measurements of atmospheric carbon: *"The new measurements are consistent with the older measurements because carbon was constantly increasing for thousands of years and now it has continued to go up, just like we expected."*
> Write a paragraph: In what ways is this person correct? In what ways is this person incorrect?

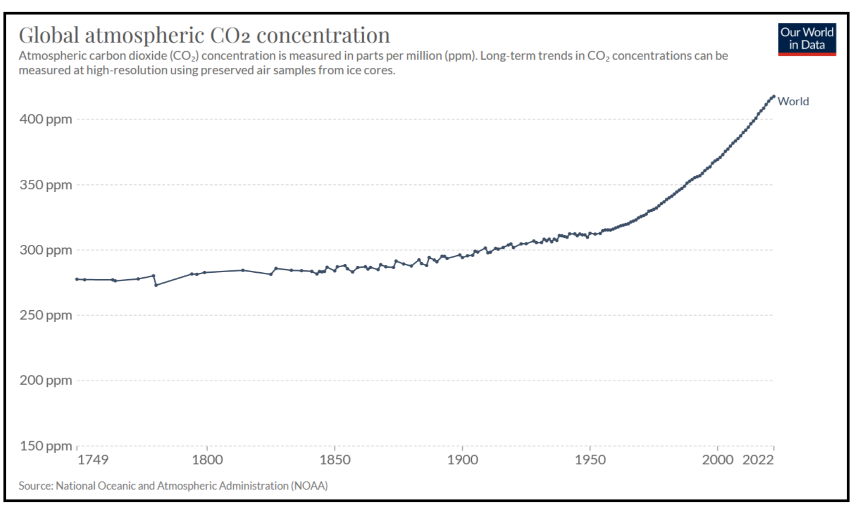
Students might note that the measurements are increasing, but the way they are increasing is different after 1850; they do not fit the linear model any longer. A student might write, "The person is correct that carbon is continuing to go up, but they are incorrect because the recent data is not the same as pattern as before. The change shown numerically in Table 1 can also be seen visually in Figure 4).

Figure 4: *NOAA atmospheric carbon data (6,000 BCE to 2022 CE) shows a drastic increase of atmospheric carbon in the most recent centuries.*



After students have noted the profound change in 1850, we asked them to either revise their historical linear model or focus on a new model for the modern era (see Figure 5). Students should consider whether the 1750–2022 data is still linear (just steeper) or whether it is a different underlying relationship, perhaps quadratic, cubic, or exponential. Using technology, various types of functions can be tried and compared to the data (OLS: A.CED.1C, F.LE.1). We felt it was worthwhile to try a cubic, for example, which seemed as though the right side can be adjusted to roughly fit the data, but it turned out to not quite have the correct "bend." Conceptually, an exponential curve makes sense, not only given the shape of the data as shown in Figure 5, but also because the addition of carbon to the atmosphere contributes to global warming and this warming can cause changes on Earth (e.g., drought reducing foliage, people using more air conditioning) that in turn lead to more carbon—a compounding effect that is the hallmark of exponential growth. This is not to say that atmospheric carbon will forever follow an exponential curve; however, the exponential increase of carbon in the atmosphere has been behaving this way for about 170 years.

Figure 5: *NOAA data on atmospheric carbon from 1749 to 2022.*



Possible Student Thinking and Discussions

Students can work to find an exponential function (OLS: F.BF.1.a.i) that reasonably models these modern data, with a strategic use of tools (OLS: F.LE.5). For example, we used Desmos to tweak coefficients until we arrived at the equation $y = 10 \cdot 1.0105^{x-1750} + 250$ with y being the amount of atmospheric carbon and x the year. Teachers can provide guidance or guidelines for elements of the model, if they wish, so that students can focus on interpretations. They should understand 250 to be a sort of foundational level of atmospheric carbon (the model will never go below that amount), they should recognize a growth rate of 1.05% each year, and the parent exponential function has been shifted to the right (to the year 1750) so that the upward curve begins in the modern era (OLS: F.IF.8.B).

Students can now contrast their new model with the earlier linear model, showing fundamental differences between linear growth and exponential growth (OLS: F.IF.4.a). Our students commented that the historical model was "slow and steady" or "pretty gradual," whereas the modern model is "steeper and steeper" or showing an increasing rate of change. Some students may say that the exponential model, before 1750, "looks" linear. This is an opportunity to point out the important difference between an exponential function when the exponent has negative input values (fairly flat) and the same function's behavior when the exponent takes on larger and larger positive input values (OLS: F.IF.7.e). We find it especially worthwhile to discuss with students how difficult it is to distinguish between a linear growth situation and an exponential growth situation if you were looking only at the "flat" portion of the exponential function (when the exponent's input values are negative). This can shed new light on Model 1 and the historical carbon data (was it exponential the whole time?), though it is also plausible that the underlying relationship changed because of the social studies context (i.e., the industrial revolution).

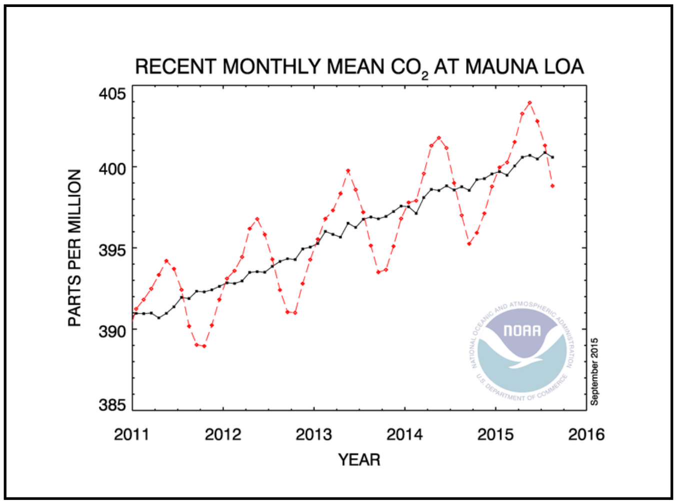
With regard to extending the model beyond the given data, students can consider whether it is possible for atmospheric carbon to continue growing exponentially (OLS: F.LE.5). It will not continue via this model forever because of physical bounds and because of some global efforts to mitigate greenhouse gases (NASA, 2022). But the problem of high levels of atmospheric carbon remains, and this exponential modeling provides compelling evidence that it is not just a continuation of normal historical variation. Our next section grapples with another sort of normal variation.

Model 3: Regular Fluctuations of Atmospheric Carbon

Task Description

The amount of atmospheric carbon is not uniform throughout the year. There is typically a variation of ±4 ppm according to the seasons. This variation can connect to science lessons on photosynthesis (OLS for Science: B.DI.3, B.C.2) and world geography (OLS for Social Studies: WG.4) as students think about the relatively large amounts of land in the northern hemisphere compared to the water-rich southern hemisphere. When plants in the northern hemisphere are active in the summer it uses up some of the atmospheric carbon, whereas the northern hemisphere's winter involves dormant plants, so the atmospheric carbon increases. Figure 6 shows this seasonal variation in atmospheric carbon. What functions could model this cyclical phenomenon?

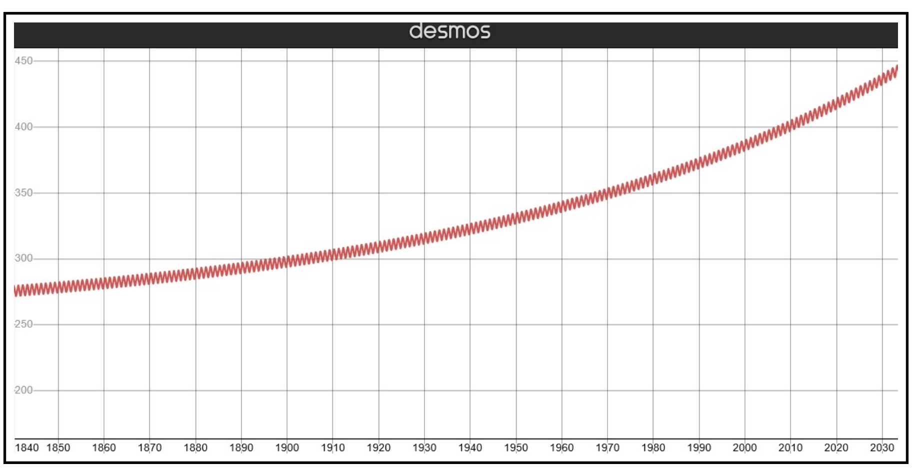
Figure 6: *NOAA data showing a regular rise and fall of atmospheric carbon (red plot) compared to the rolling average (black plot).*



Trigonometric functions are well suited to this task (OLS: F.TF.5). Focusing just on the seasonal variation (not the general upward trend), students can look for a sine function (or cosine function) that repeats its cycle every year and that goes up 4 ppm and down 4 ppm from the midline. A possibility is $y = 4sin(2\pi x)$ where x is the year and y is the amount of atmospheric carbon, as before.

As a culmination of these modeling activities, and if it is grade-level appropriate, students can be asked a final question: What is a mathematical function that represents the historical amount of atmospheric carbon and the season-to-season fluctuations? Answering this question allows students to work creatively with mathematical objects by combining multiple functions together. In particular, the seasonal variation function can be added to the historical trend function, essentially adding and subtracting the 4 ppm during winter to summer each year. The result is shown in Figure 7; if you zoom out, the historical trend is apparent, and if you zoom in, the seasonal fluctuations become more noticeable.

Figure 7: *An example of a function, in this case* $y = 10 \cdot 1.0105^{x-1750} + 250 + 4\sin(2\pi x)$*, that models the historical and the seasonal changes in atmospheric carbon over time.*



Possible Student Thinking and Discussions

For the basic trigonometric model, students can discuss how sine or cosine functions can be used to achieve the annual fluctuation, though sine is a strategic choice in this situation because the start of the calendar year corresponds with the midpoint of the increase, not the local maximum. Teachers may also choose to have students generate other examples of measurable phenomena that vary with the seasons (e.g., average monthly temperatures, number of leaves on trees), and these phenomena can be compared to atmospheric carbon with structurally similar sine models.

For the combining of functions (OLS: F.BF.1.b), students may express some hesitation. They may think that exponential functions and trigonometric functions cannot be added together because they are "different types," but because they have the same variables (atmospheric carbon in ppm as dependent upon year) it actually does make sense to add them. But we hypothesize that students may still be unsure of how to combine them, possibly thinking about composition or multiplication rather than addition. These are good conversations to have, with connections to the context. Because the output of one function is not the input of the other (rather, they have the same inputs and outputs), it does not make sense to compose them. As for multiplication, students should be encouraged to think about the context of the seasons, which literally involve taking away and adding carbon to the atmosphere which is why the seasonal fluctuations are added to the general trend. In general, we think that students' discomfort with combining functions is largely due to the fact that we do not give them many opportunities to do so; we often have them work to model just one relationship at a time.

Conclusion

As we have shared in this article, the context of climate change allows for various forms of mathematical modeling to occur, but the processes and underlying concepts of function could be assessed by asking students to work with similar ideas in a different context. For example, students could model an ice cream company that strives for annual sales growth of 3% but has natural booms in the summer and slowdowns in the winter. Teachers might also ask students to combine two functions, such as a savings account that has both compound interest (exponential) and a regular in-flow of deposits (linear). Students should recognize that these separate functions can be combined meaningfully if the independent variable and the dependent variable are consistent in both functions.

Although these mathematical ideas of modeling and combining functions can be generalized across many different contexts, we nevertheless feel that atmospheric carbon is a particularly important one. The scientific community is highly confident that human activity in the modern era has drastically increased the level of greenhouse gases, but some lay people still contend these are merely "natural" variations in the climate. The mathematical explorations described here show that the increases in atmospheric carbon since 1850 are inherently different than the increases over the previous 8,000 years, and even when we account for seasonal fluctuations, the underlying exponential growth remains apparent.

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