Exploring the Hand in Hands-On Learning: Illuminating the Standards for Mathematical Practice

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Abstract: *Exploring the diverse sizes and shapes of hands among sixth-grade students offers a practical platform for engaging with complex mathematical concepts. This study involved 113 students who investigated the area of their hands, a task that requires understanding the measurement of irregular shapes. Traditionally, middle school curricula focus on the area of regular figures, such as squares and triangles. Addressing the grades 6-8 Measurement Standard, our activity introduces methods for calculating the area of more complex shapes. We detail the instructional activities used to familiarize students with these techniques across five sixth-grade classes. The hands-on approach not only aligns with best practices in mathematics education but also enriches students' problem-solving skills.*

Keywords: *Standards for Mathematical Practice (SMP), area, inquiry, hands-on learning, non-routine problem solving*

Introduction

Are all hands the same size? Do all hands share the same shape? How do we measure the size of a hand? A wide variation of physical changes occurs among students within the "tween" years. It is particularly interesting to note the shape of student hands and the intricate composition of a body part that is uniquely specific to each individual. The size and shape of student hands may vary drastically and provide novel investigations for mathematical conjecture. While the human hand is incredibly complex, it also bears a tremendous platform to engage students in mathematical inquiry and problem solving aligned with best practices for teaching and learning mathematics.

To investigate the various sizes of intermediate students' hands, we asked 113 sixth-grade mathematics students to determine the area of their hands. To do this, students must understand the vital measurement topic of how to find the area of irregular shapes, as well as embrace the complexity of this nontrivial activity. Up until this point in time in their mathematical journey, middle school students generally have only been taught how to find the area of regular shapes, including squares, rectangles, triangles, trapezoids, parallelograms, and composite figures.

The grades 6-8 Measurement Standard of Principles and Standards for School Mathematics addresses measuring the area of complex shapes. The area of irregular shapes can be determined by a variety of methods. In this article, we describe the activities used to introduce this technique to five classes of sixth-grade mathematics students.

Learning Goals

The ultimate goal of this activity is to provide students with an authentic space to explore a nontrivial problem related to estimating the area of a hand. Our focus was for students to discover how to develop strategies to determine the area of an irregular shape that is common, yet uniquely different, to each and every student. Students were encouraged to develop their own strategy for finding the area of one of their hands, as well as to communicate and revise their strategies to classmates in a collaborative method. Throughout this activity, students engaged in the Standards for Mathematical Practice (Council of Chief State School Officers & National Governors' Association, 2022) to achieve this learning goal by focusing on the process of thinking like a mathematician.

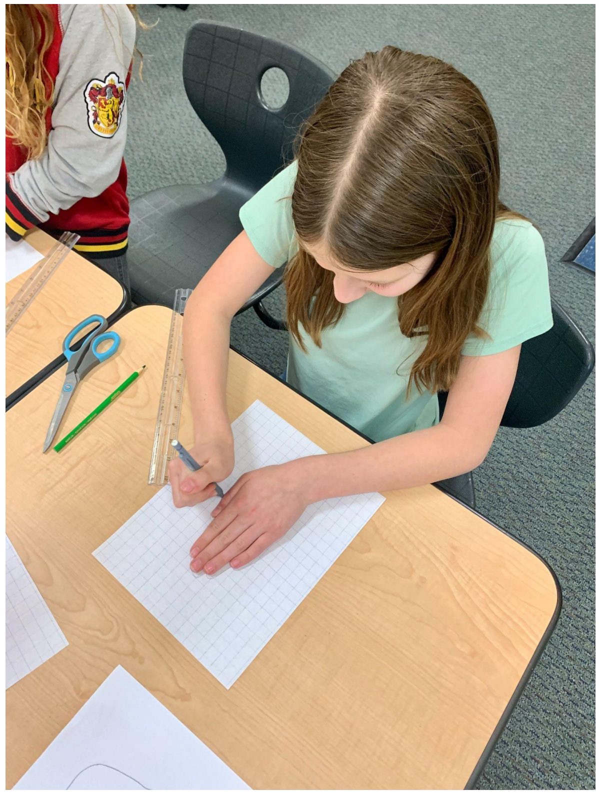
Student exploration began immediately when students were asked to consider how to most accurately determine the area of their hand. This mathematical challenge gave students a clear and explicit problem to begin making sense of. This required cognitive effort from students. Too often, students are given specific steps for how to solve math problems (example: students are shown how to complete a problem, then asked to imitate the steps with new numbers), hindering their problem-solving development. In our experience, it is hard for teachers of mathematics to "let go" of this control in cultivating thinking skills. Following the first Standard for Mathematical Practice, *Make Sense of Problems and Persevere in Solving Them*, students were supported through constructing their own methods for finding the area of their hand. Prior relevant knowledge of and experience in solving for the area of regular shapes lent a helping hand to students while they worked through this more challenging task. Students were provided time to initiate a plan themselves and then in collaboration with a small group of their fellow classmates. During the task of making sense of the problem, students were continually asked questions to guide in the development of their plan by the teacher and their classmates. Students were continuously defending and refining their ideas throughout this process.

This led students into the second Standard for Mathematical Practice, *Reason Abstractly and Quantitatively*. Students began to estimate the area of their hand. Individuality was apparent when they shared their various initial thoughts on how they would be going to go about finding their solution. The class noted that there are multiple ways to go about solving this problem, as there are for almost everything in their lives. In subsequent portrayal of the activity, students also noted some methods might be easier and more efficient than others.

The third Standard for Mathematical Practice, *Construct Viable Arguments and Critique the Reasoning of Others*, was in full effect as students shared their student-created approaches with one another. After each group had shared their method, students were encouraged to communicate their thoughts with their peers. Students were expected to provide respectful reasoning and justification for their opinions. As students compared and contrasted their various methods, an engaging mathematical discussion occurred among all students emerged as they dialogued.

The fourth Standard for Mathematical Practice, *Model with Mathematics*, finally gave students the opportunity to work with their hands (literally). As Figure 1 suggests, white copy paper and pencils were distributed to students and they immediately began to trace their hands.

Figure 1: *A student tracing a closed hand.*



It was interesting to watch as some students traced their hands with their fingers spread wide, while others eliminated the gaps between fingers as they placed their hands flat on the paper. Their eyes began to wonder as they finished tracing their hands, quickly noticing the differences between the ways they had placed their hands on the provided paper in comparison to their peers. This led to more engaging discussions on which approaches may be the most effective.

Students quickly acknowledged that they would need to be provided with more than just a blank piece of paper and a pencil to calculate the area of their hands. Thus, they were asked to create a list of items that would aid them in their problem solving. This led the class into the fifth Standard for Mathematical Practice, *Use Appropriate Tools Strategically*. To be detailed later in the article, students were then provided with the list of tools/materials they requested.

The sixth Standard for Mathematical Practice, *Attend to Precision*, was pertinent for students to obtain an accurate justification of their approach within the context of mathematical techniques. Students understood the importance of attend to precision and were sure to ask for assistance when they needed it. They were careful when tracing their hands on the provided graph paper (refer to Figure 1), as well as counting the completed and partial squares within the outline of their hand.

The seventh Standard for Mathematical Practice, *Look for and Make Use of Structure*, was observed when students began to count the squares. Students began to use techniques to work more efficiently. Some students generally drew rectangles around sections both inside and outside their traced hand to quickly calculate the number of squares within. Others used colored pencils/ crayons to expedite their calculations (refer to Figure 2).

Figure 2: *Coloring whole squares.*

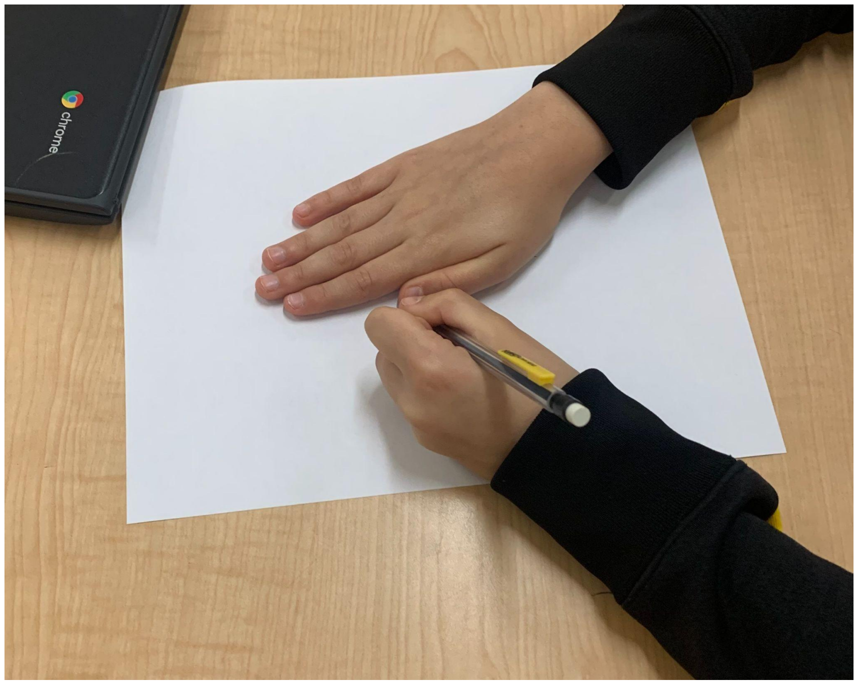
Standard 8, *Look for & Express Regularity in Repeated Reasoning*, was engaged when students began to calculate their final solution. Some students chose to count square by square while others drew rectangles around large areas to quickly multiply how many squares were found within. Still others combined these two methods. Several students expressed "ah-ha" moments as they discovered they could use their prior knowledge of multiplication to expedite their calculations rather than completing the tedious process of repeated addition.

Description of Activity/Lesson

The activity took place over two consecutive school days, during students' forty-minute mathematics 6 blocks. The first class session began with a discussion of area, led by the question "What is area?" Several students in each class volunteered to share their answers, as this was a familiar concept that had been introduced to them each year since early elementary school. As one student shared, "Area is the space inside of a shape measured in square units." Students were then asked to look at their hands and brainstorm how they could potentially find its area. They were instructed to share their thoughts with their group.

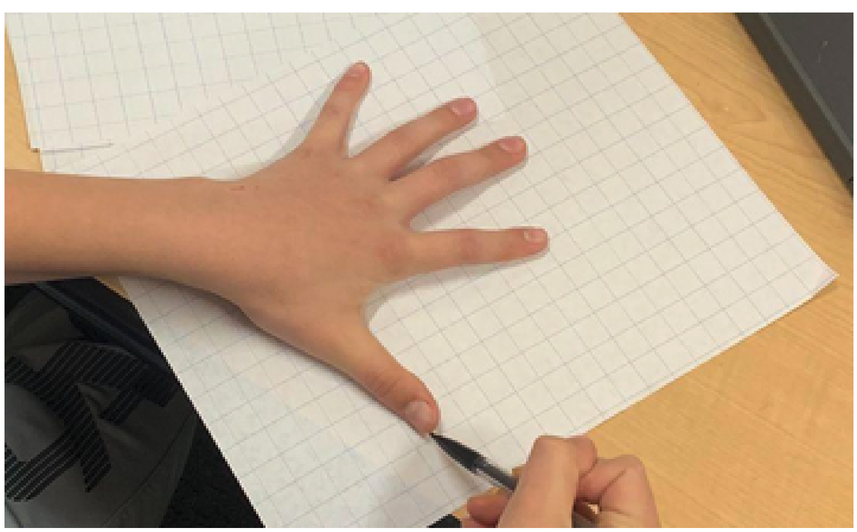
After students had time to discuss, volunteers were given the opportunity to share their ideas with the class. The first few volunteers felt passionate about attempting to measure the base and height of their hands. They recalled the formula for solving for the area of a rectangle (Area = base $\times$ height), which they felt was the shape that most closely resembled the shape of their hand. They were left puzzled when asked how they would consider the “extra space” surrounding the thinner parts of their hand. The students then began to ask for work materials. This led the class into a discussion about what tools/materials might benefit them in this process. One student quickly insisted upon the use of paper. White copy paper and a pencil were distributed to each student at this time. It was interesting to see how students immediately began to trace their hands on the provided paper in several unique ways (refer to Figure 3).

Figure 3: *A unique hand tracing.*



Some students traced their hands open (refer to Figure 4), others closed, and some attempted to alter the shape of their hands and make them more linear and rectangular.

Figure 4: *A student tracing an open hand.*



Each student was then asked to examine their hand and give their best estimate of the area. For students to provide reasonable estimates, they were each given one square-centimeter cutout, used as a unit of measure. Several students began moving the square around on their hands, as well as the tracing of their hands. Most students estimated the area of their hand to be about 60 square centimeters. Some responses were incredibly high estimates, including 112 square centimeters, while others were just the opposite with responses less than 20 square centimeters. Overall, each class determined that their estimates with the use of a single square centimeter and a piece of white multi-purpose paper were not accurate portrayals.

Students began discussing which materials and mathematical tools would help them solve the area of their hand more accurately. In the front of the classroom, as a whole group, the class began to form a list of materials they felt would make the process more manageable. This list included the following supplies: pencils, white paper, rulers, colored pencils, graph paper, square centimeters, erasers, scissors, markers, crayons, and a calculator. All students were resolute in their perceived usefulness of the graph paper to assist them in completing this task. Students then created their own list of the steps they intended to follow to solve for the area of their hands. Students' desks were arranged in groups of six. They were permitted to collaborate with their peers during a brainstorming session. Students' lists of steps include the following.

Example 1

Step 1: Trace hand on one-centimeter square graph paper.

Step 2: Count whole centimeter squares in traced hand.

Step 3: Count half centimeter squares in traced hand.

Step 4: Add all the squares and half squares together.

Step 5: Write the final answer.

Example 2

Step 1: Trace closed hand on centimeter graph paper.

Step 2: Count each whole square.

Step 3: Add the whole squares together.

Step 4: Count each half square and mark with markers.

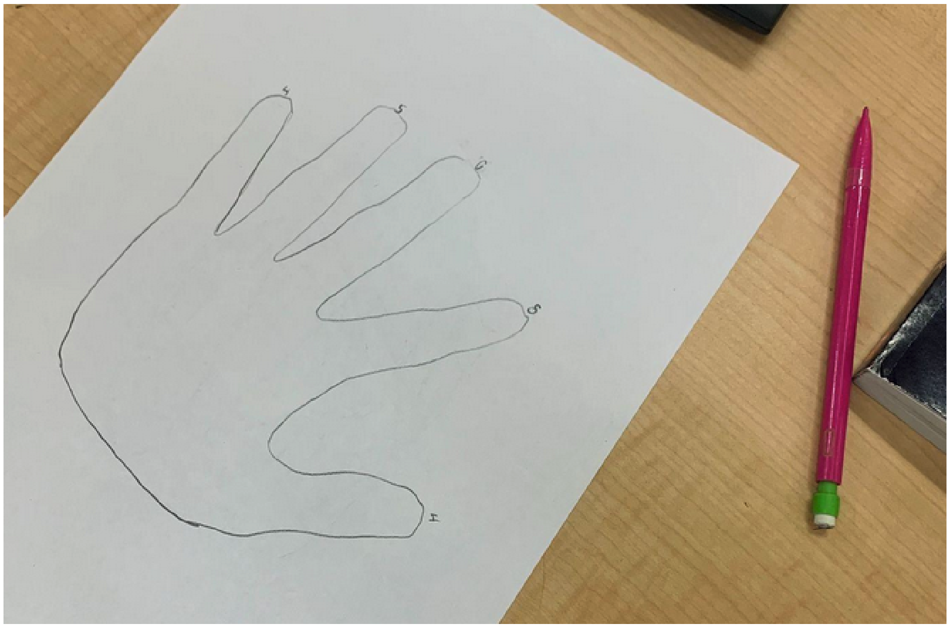
Step 5: Add each of the half squares together.

Step 6: Add the two sums together.

Step 7: Round the answer to the nearest hundredth

At the start of the second day, students were provided with the materials they requested on the previous day. Student desks remained in the same formation for this day as well. Students used the materials provided to follow the steps they had created the day prior. Many of the students followed similar steps to the ones listed above (refer to Figure 5).

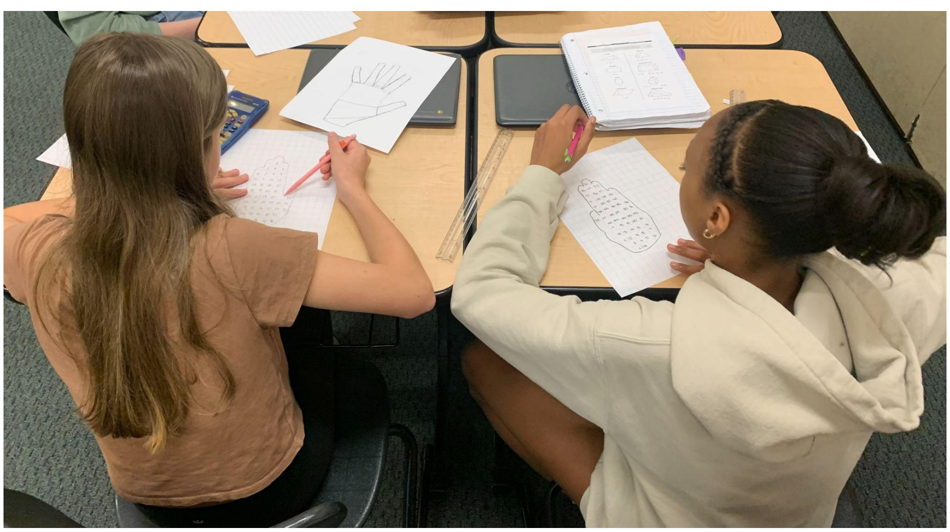
Figure 5: *Traced open hand.*



Students had unique way of counting the whole squares found within their hand. For example, some students colored the whole squares one color and the partial squares another. Others put slashes through the squares as they counted them. Several students placed small numbers inside each square, to ensure they remained organized.

As one might expect, sixth grade students had no trouble counting the whole squares found within their hands (refer to Figure 6).

Figure 6: *Counting/labeling complete squares.*



Not surprisingly, it was the partial squares that created the most challenge for them. Most of the students organized the partial squares into faction-sized categories. Some used colored pencils/markers/crayons to help them through this process. These students assigned individual colors to each of the fractional pieces that they used and colored similar-sized partial boxes the same color. For example, some students colored all the half-filled boxes red representing the fraction $\frac{1}{2}$. They had another unique color for the piece that was about $\frac{1}{4}$ the way filled, and so on (refer to Figures 7 & 8).

Figure 7: *Identifying partial squares.*

Figure 8: *Identifying partial squares.*

This technique assisted students in adding together their partial squares when they were finished adding color to each box. One student preferred to write the fraction within each partial square, so she added all the fractional parts together when she finished labeling. The methods students used to compute the total area of their hand was unique as well. Some students decided to use a calculator, while others did not. The same could be said for the use of multiplication versus the use of repeated addition. As students completed their final steps, they became eager to share their results with their peers. Students enjoyed comparing their various strategies and techniques and to see the variation of hand sizes. Students were given the opportunity to explore whether adjusting the placement of their hand and fingers on the grid paper makes this task easier or more challenging for them to conjecture about the area of their hand.

Finally, students were asked to write a short reflection on what they learned about finding the area of their hand. They were also asked to comment on how changing the grid paper scale may make their conjectures more or less accurate. One student shared, "If the grid was a different size and was bigger there would be fewer numbers to count, but the downside would be it would not be as accurate. It would be easier if the hand was closed because then you don't have to worry about the gaps between the fingers . . . if we were to do this again, one thing I would change is coloring the half to make it easier to spot and count them." This response was like several others. Another student shared, when asked if it would be easier to determine the area if their fingers were together or separate from their palm when tracing their hand, "I think it's easier to find the area with a closed traced hand because it is easier to count the squares . . . also, we do not have to worry about trying to count each individual finger." Several students also agreed with this statement.

Overall, students gained a better understanding of solving the area of irregular shapes and embracing the process-oriented elements and variability that is inherent of tasks that undergird the Standards for Mathematical Practice.

Student Conceptualizations and Approaches

Students brainstormed several unique ways to solve the problem for the area of their hands. Some were adamant that a formula must be used to solve the area of a two-dimensional shape. Others had a firmer understanding of the definition of area. It was interesting to watch as they problem-solved each step and bounced ideas off one another. As mentioned above, students immediately wanted to develop a formula to solve for the area of their hand and this is likely due to the extensive use of formulas introduced throughout the course of sixth-grade mathematics.

When students were asked to define what area is, they began to think about how the use of squares could help assist them in this process. Some students moved the square around their hand attempting to calculate how many squares it would take to fill the palm of their hand. This inevitably caused students to become frustrated with the use of their physical hands. It was at this point that they decided it would be much easier to trace their hands, this then gave them the use of both of their hands when working through the problem-solving process.

It was fascinating to watch as they each employed unique methods for tracing their hands. Each class was evenly split with the technique they used to trace their hands. About half the students placed their hands on the provided graph paper with their fingers spread wide (imagine the way you traced your hand in the elementary grades to create a hand turkey for Thanksgiving). While the remaining students traced their hands with their fingers closed tightly. This led the class into a discussion about whether this slight variation would make a difference during their problem solving. Some students expressed that the area of their hand would not change based on the placement of their hand on the graph paper. Others argued that although the area may not differ, the accuracy of the approximation may be weakened by spreading fingers out. These students felt that the best method would be to trace their hands with closed fingers. Having a whole class discussion related to a variable within the problem, such as this one, supports the spirit of the Standards for Math Practice. It also provides a scaffold to discuss more advanced concepts that may promote further investigations as an extension to the lesson.

Scaffolding Meaningful Work

Initially, when the lesson task was introduced, some students were unsure of how they were going to tackle this new challenge. Something similar among these students was their inability to think about the area in a different way. Students were so engrained with the practice of being given a formula or steps to follow that many were at a standstill when attempting to determine the area without being provided with a step-wise procedure. Up until this point, when students were taught to find the area of a new shape, they were introduced to a new formula and shown how to apply the formula around specific dimensions or constraints. This led students to believe that they needed a ruler, formula, and calculator to solve for the area of their hand (refer to Figure 9).

Figure 9: *Identifying partial squares.*

The problem they encountered was that they were not being given a formula. Some of these students began by taking measurements with their rulers. Examples of this include finding the length and width of each of their fingers, determining how wide their palm was, and doing their best to create their own formula. The variability and unknown information inherent within this activity quickly became frustrating for some students. As a point of refocusing students, they were asked to discuss their understanding of area. This encouraged student to make the connection between area and the number of squares found within the tracing of their hand.

Students were asked to determine what tools they felt might benefit them in solving this more complex task. They were slowly provided with the tools they requested, allowing them to explore various "instruments" that may be more or less valuable to them during their inquiry. This included a piece of printer paper, ruler, pencil, and a small one x one centimeter piece of grid paper. Students were given time to explore their ideas and talk with each other, gently guided by prompts and clarifying questions introduced by the students.

Some students continued to have misconceptions, as shared below. Others quickly began to make the connection that if their hand was traced on graph paper, rather than blank paper, the amount of squares they were in search of would be found within the tracing of their hand. These students were asked to share their reasoning behind requesting said graph paper. These probing questions helped refine student thinking and place them on a viable path to conjecture about the area of their hand.

Misconception Example

Step 1: Redraw our hands on graph papers.

Step 2: Cut our hands out.

Step 3: Find the area of each of your fingers using a ruler and calculator. We have to multiply the length and width together.

Step 4: Find the area of our palms with a ruler and a calculator. We have to multiply the length and width together.

Step 5: Add the areas of fingers and palms together to find the total.

The example above was common across students. Their prior experiences with area always included the use of a formula, thus many were determined to incorporate a formula within their list of steps. This reliance on a formula also stemmed from their background knowledge of solving for composite figures. This can be derived from students learning math strictly from the "I do", "we do", "you do" model of teaching without fully grasping the concept under study.

Another accuracy-based misconception that some students held was that they counted all the squares within the tracing of their hand as one square unit without acknowledging the partial squares as fractional pieces. These students were redirected and asked if they felt this was an accurate representation of the "true area" of their hand. Students seemed to grasp a better understanding that they must include the partial pieces, but a subset still struggled to count them such units meaningfully.

For many of the students, their practical pieces consisted of mostly halves. They did well identifying these pieces and adding them together. The larger and smaller pieces were the ones that seemed to cause the most confusion. Such variability was uncomfortable to students. This reality led to brainstorming and discussions on how to calculate these pieces most accurately and efficiently. Additionally, some students felt it would be best to label each piece with the fraction they felt most represented it. They then used their prior knowledge of working with fractions to then find the sum of all the fractional parts. The workshop format of the lesson delivery allowed students to continue to seek ways to revise and refine their methods. As noted previously, some students benefitted from colors to code the different particle pieces, making it easier to identify how many fractional parts.

Conclusion and Extensions

Overall, within the rich context of this complex activity, students thoroughly enjoyed this lesson. They were eager to determine the area of their hands from the moment it was introduced. Students strengthened their problem-solving abilities throughout the lesson and were able to work through initial misconceptions and redirect themselves to refine their conjectures. At the conclusion of the second day, students were asked to share something they had learned throughout the lesson. One student shared, "Something new I learned is that there are many different ways to find the area of abstract objects." Another student went on to say, "One thing I learned is that you can still find the area of a shape that is not regular." It was intriguing to watch as students strengthened their ability to problem solve and their understanding of area by processes that involved tools and communication. For this reason, the lesson was a tremendous success.

There are several ways to adapt this lesson. One suggestion to reduce one aspect of variability among the class is to distribute the same traced hand (ex. the teacher's hand) to all students, providing them with the ability to have a fixed model across the class. Once all students conjecture about the area of the hand, they will be able to more directly compare their findings with one another. It is crucial to note, as well as regularly remind students, that no one method is more or less correct than another. It is their job to continue to refine their techniques and collaborate to improve their accuracy.

It is also noteworthy to consider the size of the squares on their graph paper. With scale accounted for, students would have the opportunity to grapple with aspects of fractional pieces in a more comprehensive manner. As students make adjustments to the size of the grid paper squares they use, the need for dimensional comparison is necessary. Exploring the area of the hand around several different options of grid paper may provide another layer of discussion that supports the judicious use of tools in solving authentic problems.

References

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