An Argument for a New Description of “Division by Zero”

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Abstract: *Instead of simply characterizing division by zero as “undefined” to students, we argue that it should be considered on the basis of mathematically consistent equations. Division by zero leads to inconsistent mathematical statements that are just as invalid as other incorrect statements. We propose that division by zero should be characterized as “inconsistent” rather than “undefined.” Consistency underlies the fundamental truth of all mathematical equations and the interrelationships between mathematical objects.*

Keywords: *Division by zero, proof and argumentation, definitions*

Division by Zero

When students first encounter division by zero, they are typically told that it is “undefined." Unfortunately, many students find this characterization confusing. The main purpose of this paper is to argue that division by zero should be discussed in terms of mathematically consistent equations. As we will discuss, division by zero leads to inconsistent mathematical statements, and therefore should be characterized as “inconsistent” rather than “undefined.”

Almost everyone is familiar with the fact that division by zero is invalid. This is so well-known that for many it has become a rote phrase: “You can’t divide by zero.” To help their students make sense of division by zero, teachers often employ one or more of the following well-known demonstrations.

Multiplication and Division

One demonstration considers multiplication and division. Multiplication of numbers n (the multiplicand) and m (the multiplier) gives a product p:

$$n \cdot m = p \tag{1}$$

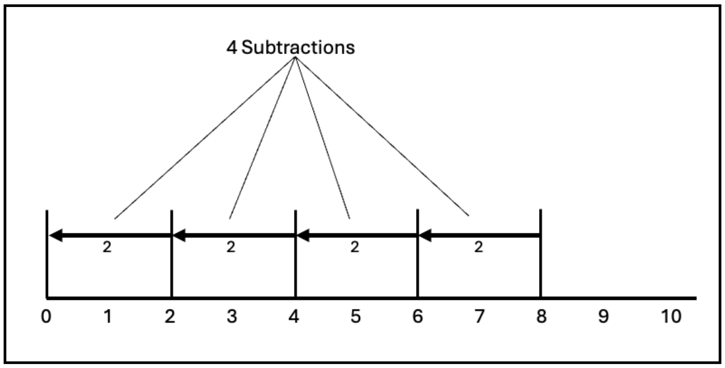
If $m = 0$, then n can be any number, since any number times zero is still zero. Equation (1) can be rewritten as p (the dividend) divided by m (the divisor) equals n (the quotient):

$$n = \frac{p}{m}, \text{where } n \text{ can be any number if } m = 0 \tag{2}$$

An Algorithmic Approach

Another demonstration considers an algorithm to calculate division of a dividend by a divisor. Here, division is calculated by the number of iterations it takes to reach 0 by repeatedly subtracting the divisor from the dividend. An example of this repeated subtraction is 8 divided by 2: $8 - 2 = 6$, $6 - 2 = 4$, $4 - 2 = 2$, $2 - 2 = 0$. A visual aid for elementary students of this example is shown in Figure 1. There are 4 iterations/subtractions necessary, so 8 divided by 2 is 4. If we were to consider a divisor of 0, a never-ending loop results. Consider 8 divided by 0. Here, 0 can be subtracted from 8 forever, and the algorithm never reaches the terminal value of 0.

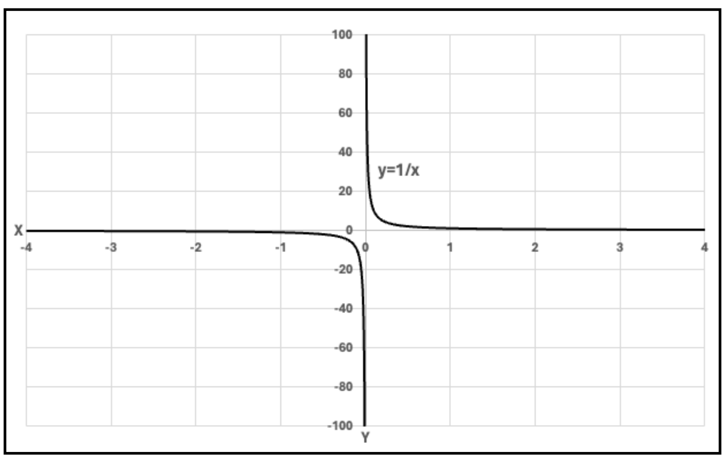
Figure 1: *A visual aid for elementary students of 8 divided by 2; a number line shows that there are 4 subtractions of 2 from 8 to get to 0.*



A Graphical Approach

A different kind of demonstration, which uses a graphical argument, can be used in more advanced classes. Here, asymptotes indicate that division by zero is infinitely large, or infinitely small (Azzolino, 2005). The plot in Figure 2 is similar to a figure shown by Azzolino for $y = \frac{1}{x}$. Here, the y axis has been expanded from $|4|$ to $|100|$ to really show the vertical asymptotes around 0. The plot was generated using Excel. Students can try to generate such a graph in Excel to really get a feel for the vertical asymptotes around 0 and how sizing the graph becomes important to realize that such asymptotes do exist no matter what the view settings of y axis are set to.

Figure 2: *A graph of $y = \frac{1}{x}$ showing vertical asymptotes around $x = 0$.*



Students can usually accept these demonstrations as showing that division by zero is invalid. However, teachers summarize the matter by saying, "Therefore we say that division by zero is undefined." While this may not have bothered all students in their formative years, others (such as this author) find it troubling. This is because an explanation about an operation involving a characteristic called "undefined" seems unsatisfying and unjustified at this level.

It seems evident that division must have first been envisioned for all divisors, and then later altered to avoid any mathematical contradictions involving division by 0. This seems justifiable. Yet, if this restriction is based on being "undefined," then it gives the impression that when any contradictions in mathematics are encountered, we would just modify the so-called "definitions" around the contradictions.

Issues with Characterizing Division by Zero as "Undefined"

Why is division by zero said to be "undefined?" What does an "undefined" operation mean? When are things defined or left undefined in mathematics, and why is this suddenly relevant to operations encountered by students?

First of all, students may find it perplexing that a specific numerical operation, division by zero, is described as "undefined" when this terminology has not been used before. Definitions may have been used previously in their mathematics education, but never in this way. The nature of a definition seems to suddenly hold more importance than it had previously. Here, the fact that something is "undefined" actually affects the validity of a simple mathematical operation.

Secondly, some students may suspect that they had not been paying careful enough attention—that they missed some important aspects or contexts for definitions. Even students who have a context for definitions ("you have to start somewhere") can feel that definitions suddenly have a higher importance in this context.

Imagine that we look up a definition in the dictionary and then proceed to look up each word in that definition (and each word in those definitions, and so on). Continuing the process over and over, we eventually find that each word is defined in terms of other words. A circular logic begins to appear. Do we ever really know any word without relying on the definition of another? It does not appear that there is a primitive word from which all others are defined. Is this due to the logic of the language, the necessity for some undefined words, the development of words that we were taught as babies or children, or our intrinsic understanding of language? Readers interested in delving deeper into this topic are encouraged to refer to Appendix A, *Additional Problems with the Term "Undefined."*

An Alternative: Consistency in Mathematical Equations

As discussed up to this point, characterizing division by zero as "undefined" is unsatisfying and perhaps hopelessly confusing to the student. Nevertheless, it is clear from the earlier demonstrations that divisibility by zero is invalid. This paper proposes a new way for discussing division by zero without the term "undefined." Division by zero will be discussed according to the consistency of mathematical equations; division by zero leads to mathematically inconsistent equations.

Any number that is divided by zero leads to a mathematical inconsistency just as inconsistent as the equation $\frac{3}{4} = 1$. Here, it is almost inherently obvious that mathematics relies on consistent and valid equations.

There is nothing special about division by zero relative to any other inconsistent equation. In fact, the whole matter for division by zero can be shown in terms of an equally inconsistent equation involving only multiplication. Consider that equation (1) uses only simple multiplication to establish an inconsistent equation involving zero. In fact, equation (1) is only a simple algebraic manipulation from equation (2), which is invalid. Any number times zero is annihilated and cannot equal a non-zero number (such as $0 \cdot x = 3$).

Note that there is no rule that "multiplication by zero is undefined" as there is for division by zero. So, we can see that anything that gives rise to an inconsistency, whether through multiplication, division, etc., must be avoided. By considering consistency, we do not need to define (or use the term "undefined") for any of the infinite number of inconsistent equations, such as division by zero. For further discussion, see *Appendix B: Consistent/Inconsistent Equations*. Certainly, consistency includes the sense that an equation must be *true* within its own context, e.g., number sense in regard to equality. The left and right sides of an equation must be equal in terms of quantities (i.e., $3 = 3$, but $3 \neq 5$).

We outline a strategy that teachers can use to introduce "division by zero" to students as follows:

1. Show some demonstrations of division by zero as inconsistent, such as examples in this paper.
2. State that division by zero is traditionally/historically described as "undefined," so that students will have the same context for this language if encountered elsewhere. Otherwise, deemphasize the term "undefined."
3. Provide the alternative for describing division by zero as "inconsistent."

Conclusion

The phrase, "division by zero is undefined", has become dead dogma. Only for historical purposes and context should teachers explain how students may hear of division by zero referred to as "undefined." Otherwise, we should avoid the use of the word "undefined" since it causes more confusion than benefit. That is, without any formal consideration of "undefined," students can easily come up with their own contexts and interpretations.

An alternative description for division by zero is proposed in terms of mathematically consistent equations. Consistent equations transcend all realms of mathematics; yet, seem so trivial and obvious that it may escape being mentioned at all. Yet, division by zero is just as invalid as any other inconsistent equation, and we need not "define" or leave "undefined" any of them. Division by zero should be explained to students as being "inconsistent."

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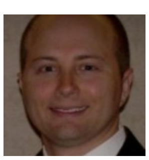

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Appendix A: Additional Problems with the Term "Undefined."

We first note that there are still conflicting and controversial views about "definitions," such as predicative definitions and impredicative definitions, in mathematical constructions (Gowers, 2008, pp. 146–147). One natural question is, "Do definitions establish mathematics?" Burton (2007) discusses how David Hilbert considered such a formalistic view of mathematics based on axioms. Ultimately Godel showed that all Hilbert's attempts would end in vain. Any set of axioms for establishing arithmetic must still rely on other undecidable propositions (Burton, 2007). So, it seems that the terms "defined" and "undefined" are in a somewhat precarious state for ensuring contradiction-free mathematics. Although mathematics may require "undefined" terms for which we could easily substitute any word, here we are speaking of an operation, specifically division by 0. We might ask whether all "undefined" concepts, such as division by zero, points, and lines, are equivalent in some way.

"Undefined" can also have different implied, intended, or comprehended meanings or contexts. Sometimes the use of the word seems to mean "non-definable"; that is, not capable of being defined, as opposed to something that is just "not defined." For example, consider the following from Kaplan (1999):

> . . . any number times zero is zero—so that $6 \cdot 0 = 0$ and $17 \cdot 0 = 0$. Hence $6 \cdot 0 = 17 \cdot 0$. If you could divide by 0, you'd get $6 \cdot \frac{0}{0} = 17 \cdot \frac{0}{0}$, the zeros would 'cancel out' (quotes mine), and 6 would equal 17. They aren't equal, so you can't legitimately divide by 0. $\frac{a}{0}$ doesn't mean anything (pp. 73–74).

Here, division by zero is stated to have no meaning. "Undefined" *stands* for something that is meaningless or "not definable." Contrast this interpretation with one from a website obtained from a quick web search of "division by zero undefined":

> 'Undefined' is defined. 'Define' means to set the limits, explain. So, undefine *means* not to set limits or not to explain (Azzolino, 2005, emphasis added).

The concluding remarks by Azzolino are even more interesting:

> There are certainly things humans can think about. Whether division by zero will in future times be defined I cannot tell . . . (Azzolino, 2005).

This interpretation goes beyond simply considering division by zero as "not defined." It leaves open the possibility for some future definition, as yet unspecified or undetermined.

Ironically, sometimes one source seems to use multiple contexts. For instance, *The World Book of Math Power* states:

> Division by zero is meaningless.
> $\frac{a}{0}$ has no answer.
>
> When zero is divided by zero no single answer is possible.
> $\frac{0}{0}$ has no one answer.
> Thus division by zero is undefined (Dell, 1996, p. 175).

In the previous passage, combining the first statement with the last suggests that "undefined" means "meaningless." Yet, for the bounded statements, "undefined" might be interpreted as the result of "no answer," "no definition," or "no unique definition." In other words, "undefined" might be something that is "not defined," "not definable," and/or "not uniquely definable."

A mathematical website, *Wolfram by Weisstein* (n.d.), discusses division by zero as follows:

> Division by zero is the operation of taking the quotient of any number x and 0, i.e., $\frac{x}{0}$. The ***uniqueness*** of division breaks down when dividing by zero, since the product $0 \cdot y = 0$ is the

> same for any y, so y cannot be recovered by inverting the process of multiplication. Zero is the only number with this property and, as a result, division by zero is ***undefined*** for real numbers and can produce a fatal condition called a "division by zero error" in computer programs (emphasis mine).
>
> To the persistent but misguided reader who insists on asking 'What happens if I do divide by zero,' Derbyshire . . . provides the slightly flippant but firm and concise response, 'You can't. It's against the rules.' Even in fields other than the real numbers, division by zero is never allowed.

Although anecdotal, the concise argument presented by Derbyshire (2004) highlights an apparent circular logic. Division by zero is not allowed because it goes against the rules. The rules are the reason, and the reason is based solely on the rules. An equally circular discussion of "undefined" occurs in the first paragraph. There, it is a result that division by zero is undefined. However, it does not seem constructive to use a result as the reason for the inability to divide by zero.

The first paragraph from Weisstein (n.d.) also explicitly states an argument for division by zero in terms of "uniqueness." Youse (1971) also identifies uniqueness in his explanation:

> Since any number times zero is zero (for example, $0 \times 6 = 0$ and $0 \times 8 = 0$), there is no *unique* number x such that $0(x) = 0$; thus $\frac{0}{0}$ is not defined. Also, since there is *no* number x such that, say, $0(x) = 6$, the quotient $\frac{6}{0}$ is not defined. In summary, division by zero is not defined. (p. 66, emphasis in original).

The multiplier, variable x, will work for any number (the solution set contains all numbers) and is not *unique*. Unfortunately, the term "unique" is arguably just as confusing. For instance, consider the equations: $0x = 0$, $3x^2 = 12$ and $0x^2 = 0$. These equations all have more than one solution for x. It may be argued that the second equation has a *finite solution set* that is unique (i.e., two solutions). However, any argument involving the *uniqueness of finite solution sets* does not appear in the text. In any case, the intended meaning for uniqueness of solutions seems to be based on the context for when it is allowed and when it is not.

Appendix B: Consistent/Inconsistent Equations

Incorrect multiplication equations are so obviously inconsistent (e.g., $0 \cdot x = 3$) that we would reject them before needing to solve explicitly for the variable (using division by zero). As indicated in Appendix A, it's also the case that some of these issues have to do with solving for an unknown variable rather than fixed equations. For example, there is nothing incorrect about the equation $0 \cdot 3 = 0$. It is only when we do not know x in $0 \cdot x = 0$ that questions arise. Notice the inconsistencies present in all of the following equations:

$$0 \cdot x = 5 \qquad 0 \cdot 5 = 1 \qquad y = \frac{5}{0} \qquad 5 = \frac{1}{0}$$

It is easy to see how discussions of consistency (and definitions etc.) descend quickly into some deeper mathematical foundations. Such a discussion for the arguments between intuition, formalism, and logic are outside the scope of this paper. But, we do note that consistency appears in some of these discussions, albeit in different contexts. For example, Hilbert considered consistency with regards to a system of axioms for geometry:

> [Hilbert] required that the system be consistent, and that the consistency of geometry could be made to depend, in his system, on that of arithmetic. He initially assumed that proving the consistency of arithmetic would not present a major obstacle and it was a long time before he realized that this was not the case (Dell, 1996, p. 139).

In Hilbert's context, consistency has to do with developing a mathematical structure that will continue to work without contradiction as the structure develops. This is similar to the use of consistency in logic, where a consistent theory does not contain a contradiction.

These calls for mathematical consistency for the purpose of preventing contradictions is similar to our call to describe division by zero within mathematics (without using the term "undefined" to confuse the subject). We can use consistency to express an underlying truth that requires no more depth than the validity of the equation itself (in terms of quantities bounding an equal sign). While a philosopher may question if 3 always equal 3, let's avoid these entanglements by making an equation true to itself in regard to the number. That is, it is only necessary to consider the truth of an equation within its own context (i.e., the quantities on both sides of an equal sign are in fact the same).